

—Chapter 4—

Renormalization and Renormalization Group

4-1 Boson Coherent States

[lecture 5.pdf \(usp.br\)](#)

[tutorial2.pdf \(weizmann.ac.il\)](#)

A. BOSON COHERENT STATES

- (1) The eigenstates of the (non-Hermitian) Fock space operators $\hat{a}_i$

$$\hat{a}_i|\phi\rangle = \phi_i|\phi\rangle$$

where

$$|\phi\rangle \equiv \prod_i e^{\phi_i \hat{a}_i^\dagger} |0\rangle = e^{\sum_i \phi_i \hat{a}_i^\dagger} |0\rangle$$

is called the boson coherent state.

PROOF:

$$\hat{a}_i|\phi\rangle = \hat{a}_i e^{\sum_j \phi_j \hat{a}_j^\dagger} |0\rangle = \hat{a}_i \prod_j e^{\phi_j \hat{a}_j^\dagger} |0\rangle = \prod_j [\hat{a}_i, e^{\phi_j \hat{a}_j^\dagger}] |0\rangle$$

Since $[\hat{a}_i, \hat{a}_j^\dagger] = 0$ as $i \neq j$

$$\begin{aligned} [\hat{a}_i, e^{\phi_i \hat{a}_i^\dagger}] &= \left[\hat{a}_i, \sum_{n=0}^{\infty} \frac{\phi_i^n}{n!} (\hat{a}_i^\dagger)^n \right] \\ &= \sum_{n=0}^{\infty} \frac{\phi_i^n}{n!} [\hat{a}_i, (\hat{a}_i^\dagger)^n] \\ &= \sum_{n=1}^{\infty} \frac{\phi_i^n}{n!} n (\hat{a}_i^\dagger)^{n-1} \\ &= \phi_i \sum_{n=1}^{\infty} \frac{\phi_i^{n-1}}{(n-1)!} (\hat{a}_i^\dagger)^{n-1} \\ &= \phi_i e^{\phi_i \hat{a}_i^\dagger} \\ \hat{a}_i|\phi\rangle &= \phi_i \prod_j e^{\phi_j \hat{a}_j^\dagger} |0\rangle = \phi_i |\phi\rangle \end{aligned}$$

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By taking the Hermitian conjugate of the coherent state, we deduce

$$\langle \phi | \hat{a}_i^\dagger = \langle \phi | \phi_i^*$$

- (2) The inner product of two Boson coherent states is given by

$$\langle \phi' | \phi \rangle = \langle 0 | \prod_k e^{\phi_k'^* \hat{a}_k} \prod_l e^{\phi_l \hat{a}_l^\dagger} | 0 \rangle = \prod_{k,l} \langle 0 | e^{\phi_k'^* \hat{a}_k} e^{\phi_l \hat{a}_l^\dagger} | 0 \rangle$$

The Baker-Campbell-Hausdorff formula for the product of the exponentials of two operators $\hat{A}$ and $\hat{B}$ is,

$$\begin{aligned} e^{\hat{A}+\hat{B}} &= e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]} \\ \langle \phi' | \phi \rangle &= \prod_{k,l} \langle 0 | e^{\phi_k'^* \hat{a}_k + \phi_l \hat{a}_l^\dagger} e^{\frac{1}{2}[\phi_k'^* \hat{a}_k, \phi_l \hat{a}_l^\dagger]} | 0 \rangle \\ &= \prod_{k,l} e^{\frac{1}{2} \phi_k'^* \phi_l \delta_{kl}} \langle 0 | e^{\phi_k'^* \hat{a}_k + \phi_l \hat{a}_l^\dagger} | 0 \rangle \\ &= \prod_k e^{\frac{1}{2} \phi_k'^* \phi_k} \langle 0 | e^{\phi_k'^* \hat{a}_k + \phi_k \hat{a}_k^\dagger} | 0 \rangle \\ &= \prod_k e^{\frac{1}{2} \phi_k'^* \phi_k} \underbrace{\langle 0 | \phi \rangle}_{=1} \\ &= e^{\frac{1}{2} \sum_k \phi_k'^* \phi_k} \end{aligned}$$

$\Rightarrow$ The set of Boson coherent states is non-orthogonal.

OS:

As a consequence of their nonorthogonality, any Boson coherent state can be expanded in terms of all the other Boson coherent states. So the Boson coherent states are not linearly independent.

- (3) The resolution of identity for the Boson coherent states reads,

$$\int \prod_i \frac{d\phi_i^* d\phi_i}{2\pi i} e^{-\sum_i |\phi_i|^2} |\phi\rangle \langle \phi| = \int d\phi^* d\phi e^{-|\phi|^2} |\phi\rangle \langle \phi| = \hat{1}$$

Thus, the set of Boson coherent states is "overcomplete".

PROOF:

The measure is:

$$\begin{aligned} \frac{d\phi_i^* d\phi_i}{2i} &= \frac{1}{2i} |J| d\Re(\phi_i) d\Im(\phi_i) \\ &= \frac{1}{2i} (2i) d\Re(\phi_i) d\Im(\phi_i) \\ &= d\Re(\phi_i) d\Im(\phi_i) \end{aligned}$$

Let ϕ in polar form be $\phi = re^{i\theta}$

$$\begin{aligned}
\frac{d\phi_i^* d\phi_i}{2i} &= r_i dr_i d\theta_i \\
\int \prod_i \frac{d\phi_i^* d\phi_i}{2\pi i} e^{-\sum_i |\phi_i|^2} |\phi\rangle\langle\phi| &= \int \prod_i \frac{r_i dr_i d\theta_i}{\pi} e^{-r_i^2} \prod_k e^{\phi_k \hat{a}_k^\dagger} |0\rangle\langle 0| \prod_l e^{\phi_l^* \hat{a}_l} \\
&= \int \prod_i \frac{r_i dr_i d\theta_i}{\pi} e^{-r_i^2} \prod_k \sum_{n_k} \frac{r_k^{n_k}}{n_k!} e^{in_k \theta_k} (\hat{a}_k^\dagger)^{n_k} |0\rangle \\
&\quad \times \langle 0| \prod_l \sum_{m_l} \frac{r_l^{m_l}}{m_l!} e^{-im_l \theta_l} (\hat{a}_l)^{m_l}
\end{aligned}$$

Considering the i -th integral, i.e., $k = l = i$ term

$$\begin{aligned}
&\frac{1}{\pi} \int r_i dr_i d\theta_i e^{-r_i^2} \sum_{n_i m_i} \frac{r_i^{n_i+m_i}}{n_i! m_i!} e^{i(n_i-m_i)\theta_i} \underbrace{(\hat{a}_i^\dagger)^{n_i} |0\rangle}_{=2\pi\delta_{n_i m_i}} \underbrace{\langle 0| (\hat{a}_i)^{m_i}}_{=2\pi\delta_{n_i m_i}} \\
&= \frac{1}{\pi} \int r_i dr_i e^{-r_i^2} \sum_{n_i m_i} \frac{r_i^{n_i+m_i}}{n_i! m_i!} \sqrt{n_i!} |n_i\rangle \langle m_i| \sqrt{m_i!} \underbrace{\int_0^{2\pi} d\theta_i e^{i(n_i-m_i)\theta_i}}_{=2\pi\delta_{n_i m_i}} \\
&= 2 \int r_i dr_i e^{-r_i^2} \sum_{n_i} \frac{r_i^{2n_i}}{n_i!} |n_i\rangle \langle n_i|
\end{aligned}$$

Since

$$\begin{aligned}
&\int r_i dr_i e^{-r_i^2} r_i^{2n_i} = \frac{n_i!}{2} \\
\Rightarrow \frac{1}{\pi} \int \frac{d\phi_i^* d\phi_i}{2i} e^{-|\phi_i|^2} &= \sum_{n_i} |n_i\rangle \langle n_i| = \hat{1} \in \mathbb{V}_i \\
\int \prod_i \frac{d\phi_i^* d\phi_i}{2\pi i} e^{-\sum_i |\phi_i|^2} |\phi\rangle\langle\phi| &= \int d\phi^* d\phi e^{-|\phi|^2} |\phi\rangle\langle\phi| \\
&= \sum_{n_1 \cdots n_i \cdots} |n_1, \cdots, n_i, \cdots\rangle \langle n_1, \cdots, n_i, \cdots| \\
&= \hat{1} \in \mathbb{W}
\end{aligned}$$

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B. FUNCTIONAL INTEGRAL REPRESENTATION OF THE PARTITION FUNCTION

(1) The partition function

$$Z = \text{Tr} e^{-\beta(\hat{H}-\mu\hat{N})} = \sum_n \langle n | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle$$

where $|n\rangle = |n_1, n_2, \dots, n_N\rangle = \hat{c}_1^\dagger \hat{c}_2^\dagger \dots \hat{c}_N^\dagger |0\rangle$ are bases in the Fock space. Insert the resolution of identity in terms of Boson coherent states,

$$\begin{aligned} Z &= \sum_n \langle n | \underbrace{\int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} |\psi\rangle \langle\psi|}_{\hat{1}} e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle \\ &= \int d\bar{\psi} d\psi e^{-\bar{\psi}\psi} \sum_n \langle n | \psi \rangle \langle \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle \end{aligned}$$

where $|\psi\rangle \equiv |\psi_1, \dots, \psi_N\rangle$

Here,

$$\langle n | \psi \rangle = \langle 0 | \hat{c}_N \dots \hat{c}_2 \hat{c}_1 | \psi \rangle = \psi_N \dots \psi_2 \psi_1 \langle 0 | \psi \rangle$$

$$\text{Let } e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle = \sum_\ell a_\ell | \ell \rangle = \sum_\ell a_\ell \hat{c}_1^\dagger \hat{c}_2^\dagger \dots \hat{c}_N^\dagger | 0 \rangle_\ell$$

Thus, we obtain

$$\begin{aligned} \langle \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle &= \langle \psi | \sum_\ell a_\ell | \ell \rangle \\ &= \sum_\ell a_\ell \langle \psi | \hat{c}_1^\dagger \hat{c}_2^\dagger \dots \hat{c}_N^\dagger | 0 \rangle_\ell \\ &= \sum_\ell a_\ell \bar{\psi}_1 \bar{\psi}_2 \dots \bar{\psi}_N \langle \psi | 0 \rangle_\ell \end{aligned}$$

Since

$$\begin{aligned} \langle \psi | 0 \rangle_\ell &= \left(\langle 0 | \prod_{i=1}^N (1 - \hat{c}_i \bar{\psi}_i) \right) | 0 \rangle_\ell = \langle 0 | 0 \rangle_\ell = 1 \\ \langle 0 | \psi \rangle &= \langle 0 | \left(\prod_{i=1}^N (1 - \psi_i \hat{c}_i^\dagger) \right) | 0 \rangle = \langle 0 | 0 \rangle = 1 \end{aligned}$$

we obtain

$$\begin{aligned} \langle \psi | e^{-\beta(\hat{H}-\mu\hat{N})} | n \rangle &= \sum_\ell a_\ell \langle \psi | \hat{c}_1^\dagger \hat{c}_2^\dagger \dots \hat{c}_N^\dagger | 0 \rangle = \sum_\ell a_\ell \bar{\psi}_1 \bar{\psi}_2 \dots \bar{\psi}_N \\ \langle n | \psi \rangle &= \langle 0 | \hat{c}_N \dots \hat{c}_2 \hat{c}_1 | \psi \rangle = \psi_N \dots \psi_2 \psi_1 \end{aligned}$$

Thus, we obtain

$$\begin{aligned}
\langle n|\psi\rangle\langle\psi|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle &= \sum_{\ell} a_{\ell}\psi_N\cdots\psi_2\psi_1\bar{\psi}_1\bar{\psi}_2\cdots\bar{\psi}_N \\
&= \sum_{\ell} a_{\ell}\psi_1\bar{\psi}_1\psi_2\bar{\psi}_2\cdots\psi_N\bar{\psi}_N \\
&= \sum_{\ell} a_{\ell}(-\bar{\psi}_1\psi_1)(-\bar{\psi}_2\psi_2)\cdots(-\bar{\psi}_N\psi_N) \\
&= \sum_{\ell} a_{\ell}(-\bar{\psi}_1)(-\bar{\psi}_2)\cdots\psi_2\psi_1 \\
&= \langle-\psi|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle\langle n|\psi\rangle
\end{aligned}$$

The partition function in terms of coherent states:

$$\begin{aligned}
Z &= \int d\bar{\psi}d\psi e^{-\bar{\psi}\psi} \sum_n \langle-\psi|e^{-\beta(\hat{H}-\mu\hat{N})}|n\rangle\langle n|\psi\rangle \\
&= \int d\bar{\psi}d\psi e^{-\bar{\psi}\psi} \langle-\psi|e^{-\beta(\hat{H}-\mu\hat{N})} \underbrace{\sum_n |n\rangle\langle n|}_{=\hat{1}} |\psi\rangle \\
&= \int d\bar{\psi}d\psi e^{-\bar{\psi}\psi} \langle-\psi|e^{-\beta(\hat{H}-\mu\hat{N})}|\psi\rangle
\end{aligned}$$

(2)

4-2 ϕ^4 -Field Theory

A. GENERATING FUNCTIONAL

- (1) Lagrangian density of a ϕ^4 theory

$$L[\phi] = \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

The action is

$$S[\phi] = S_0[\phi] + S_{\text{int}}[\phi]$$

where

$$S_0[\phi] = \int d^D x \left(\frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right) = \frac{1}{2} \int d^D x (\nabla\phi \cdot \nabla\phi + m^2\phi^2)$$

$$S_{\text{int}}[\phi] = \frac{\lambda}{4!} \int d^D x \phi^4$$

The N -point function can be written as a functional integral as

$$G^{(n)}(x_1, \dots, x_n) = \langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{\int D[\phi] \phi(x_1) \cdots \phi(x_n) e^{-S_0[\phi] - S_{\text{int}}[\phi]}}{\int D[\phi] e^{-S_0[\phi] - S_{\text{int}}[\phi]}}$$

The generating functional is

$$Z[J] = \int D[\phi] e^{-S_0[\phi] - S_{\text{int}}[\phi] + \int d^D x J(x)\phi(x)}$$

where $J(x)$ is an auxiliary external field called a source.

Thus, the N -point function is obtained from the functional derivatives,

$$G^{(n)}(x_1, \dots, x_n) = \langle \phi(x_1) \cdots \phi(x_n) \rangle = \frac{1}{Z[J]} \frac{\delta^n Z[J]}{\delta J(x_1) \cdots \delta J(x_n)} \Big|_{J=0}$$

- (2) The generating functional for the free scalar field

$$\begin{aligned} S_0[\phi] &= \frac{1}{2} \int d^D x (\nabla\phi \cdot \nabla\phi + m^2\phi^2) \\ &= \frac{1}{2} \int d^D x \phi(x) (-\nabla^2 + m^2) \phi(x) \\ &= \frac{1}{2} \int d^D x_1 d^D x_2 \phi(x_1) G_0(x_1, x_2)^{-1} \phi(x_2) \end{aligned}$$

where

$$G_0(x_1, x_2) = \langle x_1 | \frac{1}{-\nabla^2 + m^2} | x_2 \rangle = \langle \phi(x_1) \phi(x_2) \rangle_0$$

OS:

The partial integration of the gradient term gives the surface term vanishes if the ϕ -field satisfies periodic boundary conditions.

$$\begin{aligned}
Z_0[J] &= \int D[\phi] e^{-S_0[\phi] + \int d^D x J(x) \phi(x)} \\
&= \int D[\phi] e^{-\frac{1}{2} \int d^D x_1 d^D x_2 [\phi(x_1) G_0(x_1, x_2)^{-1} \phi(x_2) - J(x_1) G_0(x_1 - x_2) J(x_2)]} \\
&= \frac{1}{\sqrt{\det G_0}} e^{\frac{1}{2} \int d^D x_1 d^D x_2 J(x_1) G_0(x_1, x_2) J(x_2)}
\end{aligned}$$

- (3) The full generating functional in terms of the free field generating functional

$$\begin{aligned}
e^{-S_{\text{int}}[\phi]} &= e^{-\frac{\lambda}{4!} \int d^D x \phi^4} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!} \right)^n \int d^D x_1 \cdots d^D x_n \phi^4(x_1) \cdots \phi^4(x_n)
\end{aligned}$$

Hence, upon an expansion in powers of λ , the generating functional is

$$\begin{aligned}
Z[J] &= \int D[\phi] e^{-S_0[\phi] - S_{\text{int}}[\phi] + \int d^D x J(x) \phi(x)} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!} \right)^n \int d^D x_1 \cdots d^D x_n \phi^4(x_1) \cdots \phi^4(x_n) e^{-S_0[\phi] + \int d^D x J \phi} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!} \right)^n \int d^D x_1 \cdots d^D x_n \frac{\delta^4}{\delta J(x_1)^4} \cdots \frac{\delta^4}{\delta J(x_n)^4} Z_0[J] \\
&= e^{-\frac{\lambda}{4!} \int d^D x \frac{\delta^4}{\delta J(x)^4} Z_0[J]} \\
&= e^{-S_{\text{int}}\left[\frac{\delta}{\delta J}\right]} Z_0[J]
\end{aligned}$$

B. PERTURBATIVE EXPANSION FOR THE TWO-POINT FUNCTION

- (1) Let the two-point function $G^{(2)}(x_1 - x_2)$ as a series in the coupling constant λ ,

$$G^{(2)}(x_1 - x_2) = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} G_n^{(2)}(x_1 - x_2)$$

Using the generating functional $Z[J]$, we can write

$$G^{(2)}(x_1 - x_2) = \frac{1}{Z[J]} \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \Big|_{J=0} \dots (a)$$

where $Z[J] = e^{-S_{\text{int}}[\frac{\delta}{\delta J}]} Z_0[J]$.

Hence, the two-point function can be expressed as a ratio of two series expansions in powers of the coupling constant. Since both the numerator and the denominator have the same constant $\frac{1}{\sqrt{\det G_0}}$ that will be cancelled, we can ignore the constant $\frac{1}{\sqrt{\det G_0}}$ in the further calculations.

- (2) The denominator of Eq.(a) is given by the expansion of $Z[0]$ in powers of λ ,

$$\begin{aligned} Z[J] \Big|_{J=0} &= e^{-\frac{\lambda}{4!} \int d^D y \frac{\delta^4}{\delta J(y)^4}} Z_0[J] \Big|_{J=0} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!} \right)^n \int d^D y_1 \cdots d^D y_n \frac{\delta^4}{\delta J(y_1)^4} \cdots \frac{\delta^4}{\delta J(y_n)^4} Z_0[J] \Big|_{J=0} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!} \right)^n \int d^D y_1 \cdots d^D y_n \frac{\delta^4}{\delta J(y_1)^4} \cdots \frac{\delta^4}{\delta J(y_n)^4} \\ &\quad \times \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{1}{2} \right)^m \left[\int d^D z_1 d^D z_2 J(z_1) G_0(z_1 - z_2) J(z_2) \right]^m \end{aligned}$$

The numerator of Eq.(a) is given by an expansion in powers of the coupling constant λ ,

$$\begin{aligned} \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \Big|_{J=0} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!} \right)^n \\ &\quad \times \int d^D y_1 \cdots d^D y_n \frac{\delta^2}{\delta J(x_1) \delta J(x_2)} \frac{\delta^4}{\delta J(y_1)^4} \cdots \frac{\delta^4}{\delta J(y_n)^4} Z_0[J] \Big|_{J=0} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \left(\frac{\lambda}{4!} \right)^n \int d^D y_1 \cdots d^D y_n \frac{\delta^2}{\delta J(x_1) \delta J(x_2)} \frac{\delta^4}{\delta J(y_1)^4} \cdots \frac{\delta^4}{\delta J(y_n)^4} \\ &\quad \times \sum_{m=0}^{\infty} \frac{(-1)^m}{m!} \left(\frac{1}{2} \right)^m \left[\int d^D z_1 d^D z_2 J(z_1) G_0(z_1 - z_2) J(z_2) \right]^m \end{aligned}$$

(3) Zeroth order in λ

The denominator of Eq.(a) is simply equal to one

$$Z[0] = Z_0[0] + O(\lambda) = 1 + O(\lambda)$$

where we dropped the contribution of the determinant in $Z_0[0]$ since it does not contribute to the expectation values of the local observables.

The numerator of Eq.(a) reduces to

$$\frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \Big|_{J=0} = \frac{\delta^2 Z_0[J]}{\delta J(x_1) \delta J(x_2)} \Big|_{J=0} + O(\lambda) = G_0(x_1 - x_2) + O(\lambda)$$

Hence, we find that to zeroth order the two-point function is simply given by

$$G^{(2)}(x_1 - x_2) = G_0(x_1 - x_2) + O(\lambda)$$

(4) First order in λ

The denominator of Eq.(a) is given by

$$Z[0] = 1 + \frac{-1}{1!} \left(\frac{\lambda}{4!} \right) \int d^D y \frac{\delta^4}{\delta J(y)^4} \frac{1}{2!} \left(\frac{1}{2} \int d^D z_1 d^D z_2 J(z_1) G_0(z_1 - z_2) J(z_2) \right)^2 + O(\lambda^2)$$

OS:

Since $Z_0[J] = \left(\det(-\nabla^2 + m^2) \right)^{-1/2} e^{\frac{1}{2} \int d^D y_1 \int d^D y_2 J(y_1) G_0(y_1 - y_2) J(y_2)}$ is a bilinear functional of $J(x)$, only an even number of derivatives in $J(x)$ can be cancelled to a given order. This means that we need to expand the exponential in $Z_0[J]$ to second order.

The derivatives yield a set of delta-functions

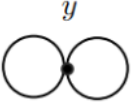
$$\frac{\delta^4}{\delta J(x)^4} [J(y_1) \cdots J(y_4)]_{J=0} = \sum_P \prod_{j=1}^4 \delta(y_{Pj} - x)$$

where P runs over the $4!$ permutations of the arguments y_1, y_2, y_3 and y_4 .

We can now write the first order correction to $Z[0]$, in the form

$$Z[0] = 1 + \frac{-1}{1!} \left(\frac{\lambda}{4!} \right) \frac{1}{2!} \left(\frac{1}{2} \right)^2 \times \int d^D y \int d^D z_1 \cdots d^D z_4 G_0(z_1 - z_2) G_0(z_3 - z_4) \sum_P \prod_{j=1}^4 \delta(z_{Pj} - y) + O(\lambda^2)$$

The contribution can be represented by a diagram



Count the number of times this picture obtained by contracting all the lines in a pairwise fashion without changing the topology of the graph. Thus, we obtain,

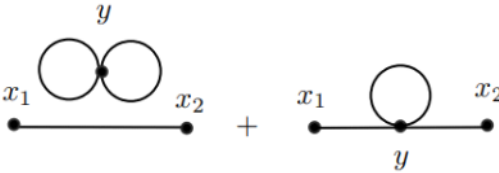
$$\begin{aligned} Z[0] &= 1 + \frac{-1}{1!} \left(\frac{\lambda}{4!} \right) S \int d^D y G_0(y, y) G_0(y, y) + O(\lambda^2) \\ &= 1 - \frac{\lambda}{8} \int d^D y \left(G_0(y, y) \right)^2 + O(\lambda^2) \end{aligned}$$

where $S = 3$ is the multiplicity factor of this diagram. Physically, the first order contribution represents corrections to the ground state energy due to vacuum fluctuations.

The numerator of Eq.(a) is given by

$$\begin{aligned} \left. \frac{\delta^2 Z[J]}{\delta J(x_1) \delta J(x_2)} \right|_{J=0} &= \frac{(-1)}{1!} \left(\frac{\lambda}{4!} \right) \int d^D y \frac{\delta^2}{\delta J(x_1) \delta J(x_2)} \frac{\delta^4}{\delta J(y)^4} Z_0[J] \Big|_{J=0} \\ &= \frac{(-1)}{1!} \left(\frac{\lambda}{4!} \right) \int d^D y \frac{\delta^2}{\delta J(x_1) \delta J(x_2)} \frac{\delta^4}{\delta J(y)^4} \\ &\quad \times \frac{1}{3!} \left(\frac{1}{2} \int d^D z_1 d^D z_2 J(z_1) G_0(z_1 - z_2) J(z_2) \right)^3 \Big|_{J=0} \end{aligned}$$

The only non-vanishing terms resulting from this process are represented by the Feynman diagrams as



The first contribution is the product of the bare propagator $G_0(x_1 - x_2)$ between the external points and the first order correction of the vacuum diagrams:

$$G_0(x_1 - x_2) \times \left[-\frac{\lambda}{8} \int d^D y \left(G_0(y, y) \right)^2 \right]$$

The second term, the "tadpole" diagram, is given by the expression

$$-\left(\frac{\lambda}{4!}\right) S \int d^D y G_0(x_1 - y) G_0(y, y) G_0(y - x_2)$$

where the multiplicity factor is $S = 4 \times 3$. It counts 12 ways of contracting the external points to the internal vertex.

By collecting terms, we get the result as

$$G^{(2)} = \frac{\text{---} + \text{---} \bigcirc \text{---} + \text{---} \bigcirc \text{---}}{1 + \bigcirc \text{---}}$$

To first order in λ , the expansion of the two-point function can also be written in the form as

$$G^{(2)} = \frac{(1 + \bigcirc \text{---} + O(\lambda^2)) \left(\text{---} + \frac{\bigcirc}{\text{---}} + O(\lambda^2) \right)}{(1 + \bigcirc \text{---} + O(\lambda^2))}$$

Thus, to first order in λ , the two-point function is given by

$$G^{(2)} = \text{---} + \frac{\bigcirc}{\text{---}} + \dots$$

$$G^{(2)}(x_1, x_2) = G_0(x_1, x_2) - \frac{\lambda}{2} \int d^D y G_0(x_1 - y) G_0(y, y) G_0(y - x_2) + O(\lambda^2)$$

OS:

The cancellation of the vacuum diagrams is a general feature of perturbation theory for all the N -point functions and known as the linked-cluster theorem.

(5) In momentum space:

$$G_0(x_1 - x_2) = \int \frac{d^D p}{(2\pi)^D} e^{i\vec{p} \cdot (x_1 - x_2)} G_0^{(2)}(p)$$

Since

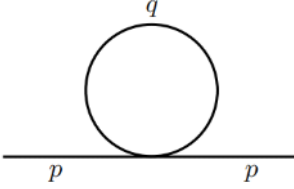
$$\begin{aligned} & \int d^D x_2 \delta^{(D)}(x_1 - x_2) (-\nabla^2 + m^2) \int \frac{d^D p}{(2\pi)^D} e^{i\vec{p} \cdot (x_1 - x_2)} G_0^{(2)}(p) \\ &= \int \frac{d^D p}{(2\pi)^D} (p^2 + m^2) e^{i\vec{p} \cdot (x_1 - x_2)} G_0^{(2)}(p) \\ &= \delta^{(D)}(x_1 - x_2) \end{aligned}$$

and

$$\int \frac{d^D p}{(2\pi)^D} e^{i\vec{p} \cdot (x_1 - x_2)} = \delta^{(D)}(x_1 - x_2)$$

$$\Rightarrow G_0^{(2)}(p) = \frac{1}{p^2 + m^2}$$

The first order contribution to the two-point function is the tadpole diagram



$$-\frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \left(\frac{1}{p^2 + m^2} \right)^2$$

The two-point function in momentum space

$$G^{(2)}(p) = G_0(p) - \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} G_0(q) \left(G_0(p) \right)^2 + O(\lambda^2)$$

$$= \frac{1}{p^2 + m^2} - \frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \left(\frac{1}{p^2 + m^2} \right)^2 + O(\lambda^2)$$

Let the self-energy function $\Sigma(p)$ be

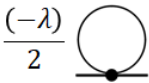
$$\Sigma(p) = -\frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} + O(\lambda^2)$$

$$\Rightarrow G^{(2)}(p) = G_0(p) + G_0(p)\Sigma(p)G_0(p) + \dots$$

$$\Rightarrow G^{(2)}(p) = G_0(p) + G_0(p)\Sigma(p)G^{(2)}(p)$$

$$\Rightarrow G^{(2)}(p) = \frac{G_0(p)}{1 - G_0(p)\Sigma(p)} = \frac{1}{G_0(p)^{-1} - \Sigma(p)}$$

To order one-loop (i.e., first order in λ or $O(\lambda)$),



$$\Sigma(p) = -\frac{\lambda}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2}$$

This expression is UV divergent for all dimensions $D > 2$, i.e., the contribution in the ultraviolet, $q \rightarrow \infty$, (or short wavelengths, high energies) is divergent.

C. PERTURBATIVE EXPANSION FOR THE FOUR-POINT

FUNCTION

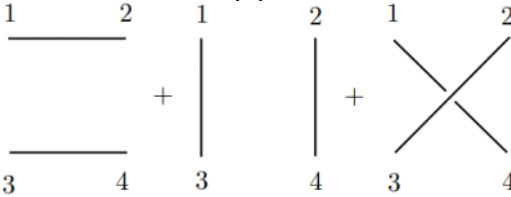
- (1) The four-point function is also known as the "two-particle" correlation function.

$$\begin{aligned}
 G^{(4)}(x_1, x_2, x_3, x_4) &= \langle \phi(x_1) \phi(x_2) \phi(x_3) \phi(x_4) \rangle \\
 &= G^{(2)}(x_1 - x_3) G^{(2)}(x_2 - x_4) + G^{(2)}(x_1 - x_2) G^{(2)}(x_3 - x_4) \\
 &\quad + G^{(2)}(x_1 - x_4) G^{(2)}(x_2 - x_3) \\
 &\quad - \int d^D y_1 \cdots d^D y_4 G^{(2)}(x_1 - y_1) G^{(2)}(x_2 - y_2) G^{(2)}(x_3 - y_3) \\
 &\quad \times G^{(2)}(x_4 - y_4) \Gamma^{(4)}(y_1, y_2, y_3, y_4)
 \end{aligned}$$

where the factors of $G^{(2)}(x_1 - x_2)$ represent the exact two-point function, and $\Gamma^{(4)}(y_1, y_2, y_3, y_4)$ is known as the four-point vertex function. The vertex function is defined as the set of 1PI diagrams.

- (2) Zeroth order in λ

$$\begin{aligned}
 G^{(4)}(x_1, x_2, x_3, x_4) &= G_0(x_1 - x_3) G_0(x_2 - x_4) + G_0(x_1 - x_2) G_0(x_3 - x_4) \\
 &\quad + G_0(x_1 - x_4) G_0(x_2 - x_3) \\
 &\quad + O(\lambda)
 \end{aligned}$$



The four-point function reduces to just products of bare two-point functions.

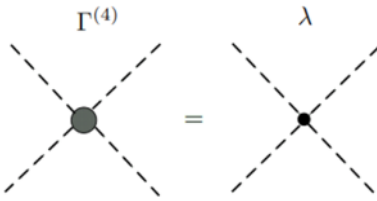
- (3) In momentum space, the vertex function $\Gamma^{(4)}$ has the form

$$\Gamma^{(4)}(p_1, p_2, p_3, p_4) = (2\pi)^D \delta^D \left(\sum_{i=1}^4 p_i \right) \bar{\Gamma}^{(4)}(p_1, p_2, p_3, p_4)$$

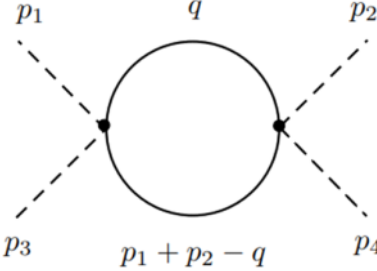
The lowest order contribution to $\bar{\Gamma}^{(4)}$ appears at order λ

$$\bar{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) = \lambda + O(\lambda^2)$$

depicted by the tree level diagrams.



To one-loop order, $O(\lambda^2)$, the four-point vertex function is



The total contribution to the vertex function $\bar{\Gamma}^{(4)}$, to order one-loop, is

$$\bar{\Gamma}^{(4)}(p_1, p_2, p_3, p_4) = \lambda - \frac{\lambda^2}{2} \left[\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2) \left((p_1 + p_2 - q)^2 + m^2 \right)} \right. \\ \left. + \text{two permutations} \right] + O(\lambda^3)$$

At the zero external momentum, i.e.,

$$\bar{\Gamma}^{(4)}(0,0,0,0) = \lambda - 3 \frac{\lambda^2}{2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2)^2} + O(\lambda^3)$$

This expression is UV divergent for $D \geq 4$.

4-3 Regularization

A. DIVERGENCE OF LOOP DIAGRAMS

- (1) The Feynman diagrams of the self-energy function $\Sigma(p)$, up to two-loop, are

$$\frac{(-\lambda)}{2} \text{ (circle on line)} + \frac{(-\lambda)^2}{4} \text{ (two circles on line)} + \frac{(-\lambda)^2}{6} \text{ (triangle on line)}$$

The Feynman diagrams of the vertex function $\bar{\Gamma}^{(4)}$, up to two-loop, are

$$-(-\lambda) \text{ (cross)} - \frac{(-\lambda)^2}{2} \text{ (circle with two crosses)} + 2 \text{ permutations}$$

$$- \frac{(-\lambda)^3}{2} \text{ (triangle with three crosses)} + 5 \text{ permutations}$$

$$- \frac{(-\lambda)^3}{4} \text{ (two circles with three crosses)} + 2 \text{ permutations}$$

$$- \frac{(-\lambda)^3}{2} \text{ (circle with a bubble and two crosses)} + 2 \text{ permutations}$$

- (2) The loop diagrams are UV divergent by dimension analysis, for examples,

$$\int \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + m^2} \rightarrow \infty$$

$$\int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 + m^2) \left((p - q)^2 + m^2 \right)} \rightarrow \infty$$

where the external momentum p is the sum of the incoming momenta:
 $p = p_1 + p_2$, $p = p_1 + p_3$, and $p = p_1 + p_4$.

- (3) To tame the UV divergence and render them finite, we need a regularization.

OS:

Regularization is a method to make the divergent integrals mathematically well-defined. This is done by introducing a cutoff or performing calculations in a $d \neq 4$ dimension where the integrals become finite.

B. UV MOMENTUM CUTOFF Λ

- (1) One-loop and two-loop diagrams of $\Sigma(p)$
 1. One-loop diagram (tadpole diagram):



$$\int_0^\Lambda \frac{d^4 q}{(2\pi)^4} \frac{1}{q^2 + m^2} = E_1(m)$$

Use the formula:

$$\int d^D q = \int q^{D-1} dq \int d\Omega_{D-1} = \frac{2\pi^{D/2}}{\Gamma(D/2)} \int q^{D-1} dq$$

OS:

$$\begin{aligned}
(\sqrt{\pi})^D &= \left(\int dx e^{-x^2} \right)^D \\
&= \int d^D x e^{-\sum_{i=1}^D x_i^2} \\
&= \int dx x^{D-1} e^{-x^2} \int d\Omega_{D-1} \\
&= \frac{1}{2} \int dx x^2 (x^2)^{D/2-1} e^{-x^2} \int d\Omega_{D-1} \\
&= \frac{1}{2} \Gamma\left(\frac{D}{2}\right) \int d\Omega_{D-1} \\
\Rightarrow \int d\Omega_{D-1} &= \frac{2(\sqrt{\pi})^D}{\Gamma(D/2)} = \frac{2\pi^{D/2}}{\Gamma(D/2)}
\end{aligned}$$

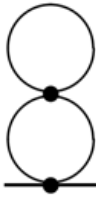
$$E_1(m) = \int_0^\Lambda \frac{q^{4-1} dq d\Omega_3}{(2\pi)^4} \frac{1}{q^2 + m^2} = \frac{1}{(2\pi)^4} \int_0^\Lambda \frac{2\pi^{4/2}}{\Gamma(4/2)} \frac{q^{4-1} dq}{q^2 + m^2}$$

Since $\Gamma(2) = 1$ and $2q dq = dq^2$

$$\begin{aligned}
E_1(m) &= \frac{1}{16\pi^2} \int_0^{\Lambda^2} dq^2 \frac{q^2}{q^2 + m^2} \\
&= \frac{m^2}{16\pi^2} \int_0^{\Lambda^2/m^2} du \frac{u}{u+1} \\
&= \frac{m^2}{16\pi^2} \int_0^{\Lambda^2/m^2} du \left(1 - \frac{1}{u+1} \right) \\
&= \frac{m^2}{16\pi^2} \left[u - \ln(u+1) \right]_0^{\Lambda^2/m^2} \\
&\approx \frac{m^2}{16\pi^2} \left[\frac{\Lambda^2}{m^2} - \ln \frac{\Lambda^2}{m^2} \right]
\end{aligned}$$

The integral is divergent for $\Lambda \rightarrow \infty$. The first term behaves like Λ^2 is called quadratically divergent and the second term behaves like $\ln \Lambda$ is called logarithmically divergent.

2. Two-loop diagram:



$$\int_0^\Lambda \frac{d^4 q_1}{(2\pi)^4} \underbrace{\frac{d^4 q_2}{(2\pi)^4} \frac{1}{q_1^2 + m^2}}_{E_1} \underbrace{\left(\frac{1}{q_2^2 + m^2} \right)^2}_{E_2} = E_1(m)E_2(m)$$

where

$$\begin{aligned} E_1(m) &= \int_0^\Lambda \frac{d^4 q_1}{(2\pi)^4} \frac{1}{q_1^2 + m^2} = \frac{m^2}{16\pi^2} \left[\frac{\Lambda^2}{m^2} - \ln \frac{\Lambda^2}{m^2} \right] \\ E_2(m) &= \int_0^\Lambda \frac{d^4 q_2}{(2\pi)^4} \frac{1}{(q_2^2 + m^2)^2} \\ &= \frac{1}{(2\pi)^4} \int_0^\Lambda \frac{2\pi^{4/2}}{\Gamma(4/2)} q_2^{4-1} dq_2 \frac{1}{(q_2^2 + m^2)^2} \\ &= \frac{1}{16\pi^2} \int_0^{\Lambda^2/m^2} du \frac{u}{(u+1)^2} \\ &= \frac{1}{16\pi^2} \int_0^{\Lambda^2/m^2} du \left(\frac{u+1}{(u+1)^2} - \frac{1}{(u+1)^2} \right) \\ &= \frac{1}{16\pi^2} \left[\ln(u+1) + \frac{1}{u+1} \right]_0^{\Lambda^2/m^2} \\ &\approx \frac{1}{16\pi^2} \left[\ln \frac{\Lambda^2}{m^2} + \frac{1}{\Lambda^2/m^2 + 1} - 1 \right] \\ &= \frac{1}{16\pi^2} \left[\ln \frac{\Lambda^2}{m^2} - \frac{\Lambda^2}{\Lambda^2 + m^2} \right] \end{aligned}$$

The integral is a logarithmically divergence for $\Lambda \rightarrow \infty$.

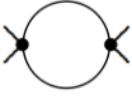
Alternative method:

$$\begin{aligned} E_2(m) &= \int_0^\Lambda \frac{d^4 q_2}{(2\pi)^4} \frac{1}{(q_2^2 + m^2)^2} \\ &= -\frac{\partial E_1(m)}{\partial m^2} \\ &= \frac{1}{16\pi^2} \ln \frac{\Lambda^2}{m^2} - \frac{\gamma}{16\pi^2} \end{aligned}$$

where γ is the Euler constant

$$\gamma = - \int_0^{\infty} dt e^{-t} \ln t = 0.5772 \dots$$

(2) One-loop diagram of $\bar{\Gamma}^{(4)}$:



$$\int_0^{\Lambda} \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 + m^2) \left((p - q)^2 + m^2 \right)} = I_1(p, m)$$

Using Feynman's parametrization:

$$\frac{1}{AB} = \frac{1}{B - A} \left(\frac{1}{A} - \frac{1}{B} \right) = \frac{1}{B - A} \int_A^B \frac{dx}{x^2} = \int_0^1 dx \frac{1}{[Ax + B(1 - x)]^2}$$

we obtain

$$\begin{aligned} I_1(p, m) &= \int_0^1 dx \int_0^{\Lambda} \frac{d^4 q}{(2\pi)^4} \frac{1}{\left[(q^2 + m^2)(1 - x) + \left((p - q)^2 + m^2 \right) x \right]^2} \\ &= \int_0^1 dx \int_0^{\Lambda} \frac{d^4 q}{(2\pi)^4} \frac{1}{[q^2 - 2pqx + p^2x + m^2]^2} \end{aligned}$$

Let $Q = q - xp$ and $M^2 = x(1 - x)p^2 + m^2$

$$\begin{aligned} I_1(p, m) &= \int_0^1 dx \int_0^{\Lambda} \frac{d^4 Q}{(2\pi)^4} \frac{1}{[Q^2 + M^2]^2} \\ &= \int_0^1 dx \frac{1}{16\pi^2} \int_0^{\Lambda^2} dQ^2 \frac{Q^2}{[Q^2 + M^2]^2} \\ &= \frac{1}{16\pi^2} \int_0^1 dx \left[\ln \frac{\Lambda^2}{M^2} - \frac{\Lambda^2}{\Lambda^2 + M^2} \right] \\ &\approx \frac{1}{16\pi^2} \int_0^1 dx \ln \frac{\Lambda^2}{x(1 - x)p^2 + m^2} \end{aligned}$$

Use the formula,

$$\int_0^1 dx \ln \left(1 + \frac{4}{a} x(1 - x) \right) = -2 + \sqrt{1 + a} \ln \frac{\sqrt{1 + a} + 1}{\sqrt{1 + a} - 1}, \quad a > 0$$

The results is then

$$\begin{aligned}
I_1(p, m) &= -\frac{1}{16\pi^2} \int_0^1 dx \ln \frac{m^2 + x(1-x)p^2}{\Lambda^2} \\
&= -\frac{1}{16\pi^2} \int_0^1 dx \left[\ln \frac{m^2}{\Lambda^2} + \ln \left(1 + \frac{4p^2}{4m^2} x(1-x) \right) \right] \\
&= \frac{1}{16\pi^2} \left[\ln \frac{\Lambda^2}{m^2} - \left(-2 + \sqrt{1 + \frac{4m^2}{p^2}} \ln \frac{\sqrt{1 + 4m^2/p^2} + 1}{\sqrt{1 + 4m^2/p^2} - 1} \right) \right] \\
&= \frac{1}{16\pi^2} \left[\ln \frac{\Lambda^2}{m^2} - L(p, m) \right]
\end{aligned}$$

where

$$L(p, m) = -2 + \sqrt{1 + \frac{4m^2}{p^2}} \ln \frac{\sqrt{1 + 4m^2/p^2} + 1}{\sqrt{1 + 4m^2/p^2} - 1}, \quad p > 0$$

- (3) The UV momentum cutoff can also be done with the help of a Gauss integral, such as e^{-q^2/Λ^2} to suppress the contributions at large momenta $q \gg \Lambda$ of the integral.

1. One-loop diagram (tadpole diagram):



$$E_1(m) = \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} e^{-\frac{q^2}{\Lambda^2}}$$

Using Feynman-Schwinger parametrization,

$$\frac{1}{A^v} = \frac{1}{\Gamma(v)} \int_0^\infty d\alpha \alpha^{v-1} e^{-\alpha A},$$

we can write

$$\begin{aligned}
E_1(m) &= \int \frac{d^D q}{(2\pi)^D} \int_0^\infty d\alpha e^{-\alpha(q^2+m^2)} e^{-\frac{q^2}{\Lambda^2}} \\
&= \int_0^\infty d\alpha e^{-\alpha m^2} \int \frac{d^D q}{(2\pi)^D} e^{-\left(\frac{1}{\Lambda^2} + \alpha\right)q^2} \\
&= \int_0^\infty d\alpha e^{-\alpha m^2} \frac{1}{(2\pi)^D} \left(\frac{\pi}{1/\Lambda^2 + \alpha} \right)^{D/2} \\
&= \frac{1}{(4\pi)^{D/2}} \int_0^\infty d\alpha e^{-\alpha m^2} \left(\frac{1}{1/\Lambda^2 + \alpha} \right)^{D/2} \\
&= \frac{1}{(4\pi)^{D/2}} \int_0^\infty d\alpha e^{-\alpha m^2} \left(\frac{m^2}{m^2/\Lambda^2 + \alpha m^2} \right)^{D/2}
\end{aligned}$$

Let $t = \frac{m^2}{\Lambda^2} + \alpha m^2$

$$\begin{aligned}
E_1(m) &= \frac{(m^2)^{D/2}}{(4\pi)^{D/2}} \int_{m^2/\Lambda^2}^\infty \frac{dt}{m^2} e^{-t+m^2/\Lambda^2} t^{-D/2} \\
&= \frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}} e^{m^2/\Lambda^2} \int_{m^2/\Lambda^2}^\infty dt t^{-D/2} e^{-t}
\end{aligned}$$

Define the incomplete gamma function

$$\Gamma(v, z) = \int_z^\infty dt t^{v-1} e^{-t}$$

Thus, we obtain

$$E_1(m) = \frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}} e^{\frac{m^2}{\Lambda^2}} \Gamma\left(1 - \frac{D}{2}, \frac{m^2}{\Lambda^2}\right)$$

For $D = 4$ and $\Lambda \rightarrow \infty$

$$E_1(m) = \frac{(m^2)^{\frac{4}{2}-1}}{(4\pi)^{4/2}} e^{\frac{m^2}{\Lambda^2}} \Gamma\left(1 - \frac{4}{2}, \frac{m^2}{\Lambda^2}\right) \approx \frac{m^2}{16\pi^2} \left[\frac{\Lambda^2}{m^2} - \ln \frac{\Lambda^2}{m^2} + \gamma \right]$$

C. DIMENSIONAL REGULARIZATION

OS:

The most common regularization method in the context of perturbation theory is dimensional regularization. Here the idea is to first calculate the integral in D dimensions and take the limit $D \rightarrow 4$ in the end.

(1) One-loop and two-loop diagrams of $\Sigma(p)$

1. One-loop diagram (tadpole diagram):



$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} = E_1(m)$$

Let $q^2 = tm^2$ and $2q dq = m^2 dt$

$$\begin{aligned} E_1(m) &= \frac{2\pi^{D/2}}{(2\pi)^D \Gamma(D/2)} \int_0^\infty q^{D-1} dq \frac{1}{q^2 + m^2} \\ &= \frac{2\pi^{D/2}}{(2\pi)^D \Gamma(D/2)} \int_0^\infty (tm^2)^{(D-1)/2} \frac{m^2 dt}{2mt^{1/2}} \frac{1}{tm^2 + m^2} \\ &= \frac{2\pi^{D/2}}{2(2\pi)^D \Gamma(D/2)} (m^2)^{D/2-1} \int_0^\infty t^{D/2-1} dt \frac{1}{t+1} \end{aligned}$$

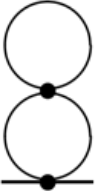
Using the Beta function

$$B(a, b) \equiv \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)} = \int_0^\infty dt t^{a-1} \frac{1}{(1+t)^{a+b}}$$

we obtain

$$\begin{aligned} E_1(m) &= \frac{2\pi^{D/2}}{2(2\pi)^D \Gamma(D/2)} (m^2)^{D/2-1} \frac{\Gamma(D/2)\Gamma(1-D/2)}{\Gamma(1)} \\ &= \frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) \end{aligned}$$

2. Two-loop diagram:



$$\int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \underbrace{\frac{1}{q_1^2 + m^2}}_{E_1} \underbrace{\left(\frac{1}{q_2^2 + m^2}\right)^2}_{E_2} = E_1(m)E_2(m)$$

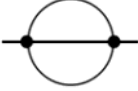
where

$$\begin{aligned}
E_2(m) &= \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{(q_2^2 + m^2)^2} \\
&= \frac{2\pi^{D/2}}{(2\pi)^D \Gamma(D/2)} \int q_2^{D-1} dq_2 \frac{1}{(q_2^2 + m^2)^2}
\end{aligned}$$

Let $q_2^2 = tm^2$ and $2q_2 dq_2 = m^2 dt$

$$\begin{aligned}
E_2(m) &= \frac{2\pi^{D/2}}{(2\pi)^D \Gamma(D/2)} \int_0^\infty (tm^2)^{(D-1)/2} \frac{m^2 dt}{2mt^{1/2}} \frac{1}{(tm^2 + m^2)^2} \\
&= \frac{2\pi^{D/2}}{2(2\pi)^D \Gamma(D/2)} (m^2)^{D/2-2} \int_0^\infty t^{D/2-1} dt \frac{1}{(t+1)^2} \\
&= \frac{2\pi^{D/2}}{2(2\pi)^D \Gamma(D/2)} (m^2)^{D/2-2} \frac{\Gamma(D/2)\Gamma(2-D/2)}{\Gamma(2)} \\
&= \frac{(m^2)^{D/2-2} \Gamma\left(2-\frac{D}{2}\right)}{(4\pi)^{D/2} \Gamma(2)} \\
&= \frac{(m^2)^{D/2-2}}{(4\pi)^{D/2}} \Gamma\left(2-\frac{D}{2}\right)
\end{aligned}$$

3. Two-loop diagram (sunset diagram):



$$\int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{1}{q_1^2 + m^2} \underbrace{\frac{1}{q_2^2 + m^2} \frac{1}{(p - q_1 - q_2)^2 + m^2}}_{I_1} = E_3(p, m)$$

where

$$E_3(p, m) = \int \frac{d^D q_1}{(2\pi)^D} \frac{1}{q_1^2 + m^2} I_1(p - q_1, m)$$

Here,

$$I_1(p - q_1, m) = \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{q_2^2 + m^2} \frac{1}{(p - q_1 - q_2)^2 + m^2}$$

Using Feynman's parametric integral formula:

$$\begin{aligned}
\frac{1}{AB} &= \frac{1}{B-A} \left(\frac{1}{A} - \frac{1}{B} \right) \\
&= \frac{1}{B-A} \int_A^B \frac{dx}{x} \\
&= \int_0^1 dx \frac{1}{[Ax + B(1-x)]^2}
\end{aligned}$$

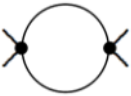
we obtain

$$\begin{aligned}
I_1(p - q_1, m) &= \int_0^1 dx \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{\left[(q_2^2 + m^2)(1-x) + \left((p - q_1 - q_2)^2 + m^2 \right) x \right]^2} \\
&= \int_0^1 dx \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{\left[q^2 - 2(p - q_1)q_2 x + (p - q_1)^2 x + m^2 \right]^2} \\
&= \frac{1}{(4\pi)^{D/2}} \frac{\Gamma\left(2 - \frac{D}{2}\right)}{\Gamma(2)} \int_0^1 dx \frac{1}{\left[(p - q_1 - q_2)^2 x(1-x) + m^2 \right]^{2-D/2}} \\
&= \frac{1}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx \frac{1}{\left[(p - q_1 - q_2)^2 x(1-x) + m^2 \right]^{2-D/2}}
\end{aligned}$$

The calculation is rather difficult, because a naive introduction of the parameter integrals results in divergences of the parameter integral. The problem is solved by lowering the degree of divergence via partial integration.

(2) One-loop and two-loop diagrams of $\bar{\Gamma}^{(4)}(p)$

1. One-loop diagram:



$$\int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2) \left((p - q)^2 + m^2 \right)} = I_1(p, m)$$

where the external momentum p is the sum of the incoming momenta: $p = p_1 + p_2$.

Using Feynman's parametric integral formula:

$$\begin{aligned}
\frac{1}{AB} &= \frac{1}{B-A} \left(\frac{1}{A} - \frac{1}{B} \right) \\
&= \frac{1}{B-A} \int_A^B \frac{dx}{x} \\
&= \int_0^1 dx \frac{1}{[Ax + B(1-x)]^2}
\end{aligned}$$

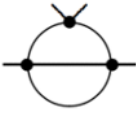
we obtain

$$\begin{aligned}
I_1(p, m) &= \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2) \left((p-q)^2 + m^2 \right)} \\
&= \int_0^1 dx \int \frac{d^D q}{(2\pi)^D} \frac{1}{\left[(q^2 + m^2)(1-x) + \left((p-q)^2 + m^2 \right) x \right]^2} \\
&= \int_0^1 dx \int \frac{d^D q}{(2\pi)^D} \frac{1}{[q^2 - 2pqx + p^2x + m^2]^2} \\
&= \frac{1}{(4\pi)^{D/2}} \frac{\Gamma\left(2 - \frac{D}{2}\right)}{\Gamma(2)} \int_0^1 dx \frac{1}{[p^2x(1-x) + m^2]^{2-D/2}} \\
&= \frac{1}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx \frac{1}{[p^2x(1-x) + m^2]^{2-D/2}}
\end{aligned}$$

NOTE:

$$I_1(0, m) = \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2)(q^2 + m^2)} = E_2(m)$$

2. Two-loop diagram:

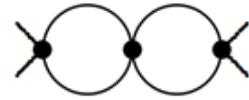


$$\begin{aligned}
& \int \frac{d^D q_1}{(2\pi)^D} \frac{1}{(q_1^2 + m^2) \left((p_1 + p_2 - q_1)^2 + m^2 \right)} \\
& \quad \times \underbrace{\int \frac{d^D q_2}{(2\pi)^D} \frac{1}{(q_2^2 + m^2) \left((p_3 + q_1 - q_2)^2 + m^2 \right)}}_{I_1} \\
& = \int \frac{d^D q_1}{(2\pi)^D} \frac{1}{(q_1^2 + m^2) \left((p_1 + p_2 - q_1)^2 + m^2 \right)} \times I_1(p_3 + q_1, m)
\end{aligned}$$

where

$$\begin{aligned}
& I_1(p_3 + q_1, m) \\
& = \frac{1}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx \frac{1}{\left[(p_3 + q_1)^2 x(1-x) + m^2 \right]^{2-D/2}}
\end{aligned}$$

3. Two-loop diagram:



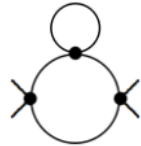
$$\begin{aligned}
& \int \frac{d^D q_1}{(2\pi)^D} \frac{1}{(q_1^2 + m^2) \left((p - q_1)^2 + m^2 \right)} \\
& \quad \times \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{(q_2^2 + m^2) \left((p - q_2)^2 + m^2 \right)} = I_1(p, m) \times I_1(p, m)
\end{aligned}$$

where

$$I_1(p, m) = \frac{1}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx \frac{1}{\left[p^2 x(1-x) + m^2 \right]^{2-D/2}}$$

Here, p denotes either of the three different momentum combinations $p_1 + p_2$, $p_1 + p_3$, and $p_1 + p_4$.

4. Two-loop diagram:



$$\int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \underbrace{\frac{1}{q_1^2 + m^2}}_{E_1} \underbrace{\frac{1}{(q_2^2 + m^2)^2} \frac{1}{(p - q_2)^2 + m^2}}_{I_2} = E_1(m) I_2(p, m)$$

where

$$I_2(p, m) = \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{(q_2^2 + m^2)^2} \frac{1}{(p - q_2)^2 + m^2}$$

Using Feynman's parametric integral formula,

$$\frac{1}{A^a B^b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \int_0^1 dx \frac{(1-x)^{a-1} x^{b-1}}{[Ax + B(1-x)]^{a+b}}$$

we obtain

$$\begin{aligned} I_2(p, m) &= \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} \int_0^1 dx \int \frac{d^D q_2}{(2\pi)^D} \\ &\quad \times \frac{1-x}{\left[(q_2^2 + m^2)(1-x) + ((p - q_2)^2 + m^2)x \right]^3} \\ &= \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} \int_0^1 dx \int \frac{d^D q_2}{(2\pi)^D} \frac{1-x}{[q_2^2 - 2pq_2x + p^2x + m^2]^3} \\ &= \frac{\Gamma(3)}{\Gamma(2)\Gamma(1)} \frac{1}{(4\pi)^{D/2}} \frac{\Gamma\left(3 - \frac{D}{2}\right)}{\Gamma(3)} \int_0^1 dx \frac{1-x}{[p^2x(1-x) + m^2]^{3-D/2}} \\ &= \frac{1}{(4\pi)^{D/2}} \Gamma\left(3 - \frac{D}{2}\right) \int_0^1 dx \frac{1-x}{[p^2x(1-x) + m^2]^{3-D/2}} \end{aligned}$$

D. ϵ -EXPANSION OF THE GAMMA FUNCTION

(1) The ϵ -expansion of the Gamma function:

For any integer n :

$$\Gamma(-n + \epsilon)$$

$$= \frac{(-1)^n}{n!} \left[\frac{1}{\epsilon} + \psi(n+1) + \frac{\epsilon}{2} \left(\frac{\pi^2}{3} + \psi(n+1)^2 - \psi'(n+1) \right) \right] + O(\epsilon^2)$$

where ψ is the digamma function,

$$\psi(n) = \frac{\Gamma'(n)}{\Gamma(n)}$$

$$\psi'(n) = \frac{\Gamma''(n)}{\Gamma(n)} - \left(\frac{\Gamma'(n)}{\Gamma(n)} \right)^2$$

OS:

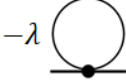
Appendix 8D Expansion of Gamma Function, P126-P128.

Kleinert, Hagen & Schulte-Frohlinde, Verena, *Critical*

- (2) One-loop and two-loop diagrams of $\Sigma(p)$ in $D = 4 - \epsilon$ dimension:

Introduce a dimensionless coupling constant $g = \lambda\mu^{D-4} = \lambda\mu^{-\epsilon}$, where μ is an arbitrary mass parameter.

1. One-loop diagram (tadpole diagram):



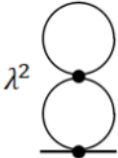
$$-\lambda \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} = -\lambda \frac{(m^2)^{D/2-1}}{(4\pi)^{D/2}} \Gamma\left(1 - \frac{D}{2}\right) = -\lambda E_1(m)$$

where

$$\begin{aligned} -\lambda E_1(m) &= -\mu^\epsilon g \frac{(m^2)^{\frac{4-\epsilon}{2}-1}}{(4\pi)^{\frac{4-\epsilon}{2}}} \Gamma\left(1 - \frac{(4-\epsilon)}{2}\right) \\ &= -\frac{m^2 g}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2} \Gamma\left(-1 + \frac{\epsilon}{2}\right) \\ &= -\frac{m^2 g}{(4\pi)^2} e^{\ln\left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2}} \Gamma\left(-1 + \frac{\epsilon}{2}\right) \\ &= -\frac{m^2 g}{(4\pi)^2} \left[1 + \frac{\epsilon}{2} \ln \frac{4\pi\mu^2}{m^2} + O(\epsilon^2)\right] \frac{(-1)^1}{1!} \left[\frac{2}{\epsilon} + \psi(2) + O(\epsilon)\right] \\ &\approx \frac{m^2 g}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(2) + \ln \frac{4\pi\mu^2}{m^2}\right] \end{aligned}$$

The residue of the pole is proportional to m^2 and independent of μ^2 . The Feynman integral is UV-divergent in four dimension, which is reflected by poles in the Gamma function at $D = 4$.

2. Two-loop diagram:



$$\lambda^2 \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \underbrace{\frac{1}{q_1^2 + m^2}}_{E_1} \underbrace{\left(\frac{1}{q_2^2 + m^2}\right)^2}_{E_2} = (-\lambda)E_1(m) \times (-\lambda)E_2(m)$$

where

$$-\lambda E_1(m) = \frac{m^2 g}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(2) + \ln \frac{4\pi\mu^2}{m^2} \right]$$

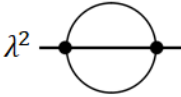
The integral over q_2 is

$$\begin{aligned} -\lambda E_2(m) &= -\lambda \frac{(m^2)^{D/2-2}}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) \\ &= -\mu^\epsilon g \frac{(m^2)^{\frac{4-\epsilon}{2}-2}}{(4\pi)^{\frac{4-\epsilon}{2}}} \Gamma\left(2 - \frac{(4-\epsilon)}{2}\right) \\ &= -\frac{g}{(4\pi)^2} \left(\frac{4\pi\mu^2}{m^2}\right)^{\epsilon/2} \Gamma\left(\frac{\epsilon}{2}\right) \\ &= -\frac{g}{(4\pi)^2} \left[1 + \frac{\epsilon}{2} \ln \frac{4\pi\mu^2}{m^2} + O(\epsilon^2)\right] \frac{(-1)^0}{0!} \left[\frac{2}{\epsilon} + \psi(1) + O(\epsilon)\right] \\ &= -\frac{g}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(1) + \ln \frac{4\pi\mu^2}{m^2}\right] \end{aligned}$$

The Feynman integral is

$$\begin{aligned} \lambda^2 E_1(m) E_2(m) &= \frac{m^2 g}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(2) + \ln \frac{4\pi\mu^2}{m^2}\right] \\ &\quad \times (-1) \frac{g}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(1) + \ln \frac{4\pi\mu^2}{m^2}\right] \\ &= -\frac{m^2 g^2}{(4\pi)^4} \left[\frac{4}{\epsilon^2} + 2 \frac{\psi(1) + \psi(2)}{\epsilon} + \frac{4}{\epsilon} \ln \frac{4\pi\mu^2}{m^2}\right] \end{aligned}$$

3. Two-loop diagram (sunset diagram):



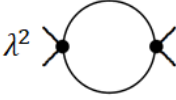
$$\lambda^2 \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \frac{1}{q_1^2 + m^2} \underbrace{\frac{1}{q_2^2 + m^2} \frac{1}{(p - q_1 - q_2)^2 + m^2}}_{I_1} = \lambda^2 E_3(p, m)$$

where

$$\begin{aligned}
\lambda^2 E_3(p, m) &= \lambda^2 \int \frac{d^D q_1}{(2\pi)^D} \frac{1}{q_1^2 + m^2} I_1(p - q_1, m) \\
&= -\frac{m^2 g^2}{(4\pi)^4} \left[\frac{6}{\epsilon^2} + \frac{6}{\epsilon} \left(\frac{3}{2} + \psi(1) + \ln \frac{4\pi\mu^2}{m^2} \right) + \frac{p^2}{2m^2\epsilon} \right] \\
&= -\frac{m^2 g^2}{(4\pi)^4} \left[\frac{6}{\epsilon^2} + \frac{18}{2\epsilon} + \frac{6\psi(1)}{\epsilon} + \frac{6}{\epsilon} \ln \frac{4\pi\mu^2}{m^2} + \frac{p^2}{2m^2\epsilon} \right]
\end{aligned}$$

(3) One-loop and two-loop diagrams of $\bar{\Gamma}^{(4)}(p)$ in $D = 4 - \epsilon$ dimension:

1. One-loop diagram:



$$\lambda^2 \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 + m^2) \left((p - q)^2 + m^2 \right)} = \lambda^2 I_1(p, m)$$

where the external momentum p is the sum of the incoming momenta: $p = p_1 + p_2$.

Thus, we obtain

$$\begin{aligned}
\lambda^2 I_1(p, m) &= \lambda^2 \frac{1}{(4\pi)^{D/2}} \Gamma\left(2 - \frac{D}{2}\right) \int_0^1 dx \frac{1}{[p^2 x(1-x) + m^2]^{2 - \frac{D}{2}}} \\
&= \frac{\mu^{2\epsilon} g^2}{(4\pi)^{\frac{4-\epsilon}{2}}} \Gamma\left(2 - \frac{4-\epsilon}{2}\right) \int_0^1 dx \frac{1}{[p^2 x(1-x) + m^2]^{2 - \frac{4-\epsilon}{2}}} \\
&= \frac{\mu^\epsilon g^2}{(4\pi)^2} \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 dx \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right)^{\epsilon/2} \\
&= \frac{\mu^\epsilon g^2}{(4\pi)^2} \frac{(-1)^0}{0!} \left[\frac{2}{\epsilon} + \psi(1) + O(\epsilon) \right] \\
&\quad \times \left[1 + \frac{\epsilon}{2} \int_0^1 dx \ln \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right) + O(\epsilon^2) \right] \\
&\approx \frac{\mu^\epsilon g^2}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(1) + \int_0^1 dx \ln \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right) \right]
\end{aligned}$$

The divergence for $D = 4$ is contained in the Gamma function which possesses poles at $D = 4$. The remaining parameter integral is finite for any D as long as $m^2 \neq 0$.

Use the formula,

$$\int_0^1 dx \ln \left(1 + \frac{4}{a} x(1-x) \right) = -2 + \sqrt{1+a} \ln \frac{\sqrt{1+a} + 1}{\sqrt{1+a} - 1}, \quad a > 0$$

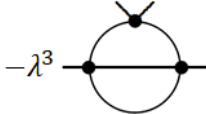
The result is then

$$\begin{aligned} & \int_0^1 dx \ln \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right) \\ &= - \int_0^1 dx \ln \left(\frac{p^2 x(1-x) + m^2}{4\pi\mu^2} \right) \\ &= \ln \frac{4\pi\mu^2}{m^2} - \int_0^1 dx \ln \left(1 + \frac{4p^2}{4m^2} x(1-x) \right) \\ \text{Let } L(p, m) &= \begin{cases} -2 + \sqrt{1 + \frac{4m^2}{p^2}} \ln \frac{\sqrt{1 + 4m^2/p^2} + 1}{\sqrt{1 + 4m^2/p^2} - 1}, & p > 0 \\ 0, & p = 0 \end{cases} \end{aligned}$$

Thus, we obtain

$$\lambda^2 I_1(p, m) = \frac{\mu^\epsilon g^2}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(1) + \ln \frac{4\pi\mu^2}{m^2} - L(p, m) \right]$$

2. Two-loop diagram:



$$\begin{aligned} & -\lambda^3 \int \frac{d^D q_1}{(2\pi)^D} \frac{d^D q_2}{(2\pi)^D} \underbrace{\frac{1}{q_1^2 + m^2}}_{E_1} \underbrace{\frac{1}{(q_2^2 + m^2)^2} \frac{1}{(p - q_2)^2 + m^2}}_{I_2} \\ &= (-\lambda) E_1(m) \times \lambda^2 I_2(p, m) \end{aligned}$$

where

$$-\lambda E_1(m) = \frac{m^2 g}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(2) + \ln \frac{4\pi\mu^2}{m^2} \right]$$

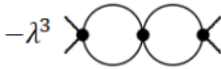
We have

$$\begin{aligned}
\lambda^2 I_2(p, m) &= \lambda^2 \frac{1}{(4\pi)^{D/2}} \Gamma\left(3 - \frac{D}{2}\right) \int_0^1 dx \frac{1-x}{[p^2 x(1-x) + m^2]^{3-D/2}} \\
&= \frac{\mu^{2\epsilon} g^2}{(4\pi)^{2-\epsilon/2}} \Gamma\left(1 + \frac{\epsilon}{2}\right) \int_0^1 dx \frac{1-x}{[p^2 x(1-x) + m^2]^{1+\epsilon/2}} \\
&= \frac{\mu^\epsilon g^2}{(4\pi)^2} \left[1 + \frac{\epsilon}{2} \ln 4\pi\mu^2 + O(\epsilon^2)\right] \frac{\epsilon}{2} \Gamma\left(\frac{\epsilon}{2}\right) \\
&\quad \times \int_0^1 dx \frac{1-x}{p^2 x(1-x) + m^2} \\
&\approx \frac{\mu^\epsilon g^2}{(4\pi)^2} \left[1 + \frac{\epsilon}{2} \ln 4\pi\mu^2 + O(\epsilon^2)\right] \times \frac{\epsilon}{2} \times \left[\frac{2}{\epsilon} + \psi(1) + O(\epsilon)\right] \\
&\quad \times \int_0^1 dx \frac{1-x}{p^2 x(1-x) + m^2} \\
&= \frac{\mu^\epsilon g^2}{(4\pi)^2} \int_0^1 dx \frac{1-x}{p^2 x(1-x) + m^2}
\end{aligned}$$

The Feynman integral is

$$\begin{aligned}
&-\lambda^3 E_1(m) I_2(p, m) \\
&= \frac{m^2 g}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(2) + \ln \frac{4\pi\mu^2}{m^2} \right] \times \frac{\mu^\epsilon g^2}{(4\pi)^2} \int_0^1 dx \frac{1-x}{p^2 x(1-x) + m^2} \\
&= \frac{\mu^\epsilon g^3}{(4\pi)^4} \frac{2}{\epsilon} \int_0^1 dx \frac{m^2(1-x)}{p^2 x(1-x) + m^2}
\end{aligned}$$

3. Two-loop diagram:



$$\begin{aligned}
&-\lambda^3 \int \frac{d^D q_1}{(2\pi)^D} \frac{1}{(q_1^2 + m^2) \left((p - q_1)^2 + m^2 \right)} \\
&\quad \times \int \frac{d^D q_2}{(2\pi)^D} \frac{1}{(q_2^2 + m^2) \left((p - q_2)^2 + m^2 \right)} \\
&= -\frac{1}{\lambda} \lambda^2 I_1(p, m) \times \lambda^2 I_1(p, m)
\end{aligned}$$

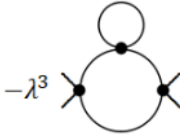
where

$$\lambda^2 I_1(p, m) = \frac{\mu^\epsilon g^2}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(1) + \int_0^1 dx \ln \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right) \right]$$

The Feynman integral is

$$\begin{aligned}
& \lambda^3 I_1(p, m)^2 \\
&= -\frac{1}{\mu^\epsilon g} \left(\frac{\mu^\epsilon g^2}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(1) + \int_0^1 dx \ln \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right) \right] \right)^2 \\
&= -\frac{1}{\mu^\epsilon g} \frac{\mu^{2\epsilon} g^4}{(4\pi)^2} \left[\frac{4}{\epsilon^2} + \frac{4}{\epsilon} \psi(1) + \frac{4}{\epsilon} \int_0^1 dx \ln \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right) \right] \\
&= -\frac{\mu^\epsilon g^3}{(4\pi)^4} \left[\frac{4}{\epsilon^2} + \frac{4}{\epsilon} \psi(1) + \frac{4}{\epsilon} \int_0^1 dx \ln \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right) \right] \\
&= -\frac{\mu^\epsilon g^3}{(4\pi)^4} \left[\frac{4}{\epsilon^2} + \frac{4}{\epsilon} \psi(1) + \frac{4}{\epsilon} \left(\ln \frac{4\pi\mu^2}{m^2} - L(p, m) \right) \right], \quad p > 0
\end{aligned}$$

4. Two-loop diagram:



Let the external momentum p be the sum of the incoming momenta: $p = p_1 + p_2$.

$$\begin{aligned}
& -\lambda^3 \int \frac{d^D q_1}{(2\pi)^D} \frac{1}{(q_1^2 + m^2) \left((p - q_1)^2 + m^2 \right)} \\
& \quad \times \underbrace{\int \frac{d^D q_2}{(2\pi)^D} \frac{1}{(q_2^2 + m^2) \left((p_3 + q_1 - q_2)^2 + m^2 \right)}}_{I_1} \\
&= -\lambda^3 \int \frac{d^D q_1}{(2\pi)^D} \frac{1}{(q_1^2 + m^2) \left((p - q_1)^2 + m^2 \right)} \times I_1(p_3 + q_1, m) \\
&= \frac{\mu^\epsilon g^3}{(4\pi)^4} \frac{2}{\epsilon^2} \left[1 + \frac{\epsilon}{2} + \epsilon \psi(1) + \epsilon \int_0^1 dx \ln \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right) \right] \\
&= \frac{\mu^\epsilon g^3}{(4\pi)^4} \left[\frac{2}{\epsilon^2} + \frac{1}{\epsilon} + \frac{2\psi(1)}{\epsilon} + \frac{2}{\epsilon} \int_0^1 dx \ln \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right) \right] \\
&= \frac{\mu^\epsilon g^3}{(4\pi)^4} \left[\frac{2}{\epsilon^2} + \frac{1}{\epsilon} + \frac{2\psi(1)}{\epsilon} + \frac{2}{\epsilon} \left(\ln \frac{4\pi\mu^2}{m^2} - L(p, m) \right) \right], \quad p > 0
\end{aligned}$$

(4) After dimensional regularization in $4 - \epsilon$ dimensions, those Feynman

integrals diverge in a specific way for $\epsilon \rightarrow 0$. For an increasing number of loops L , the Feynman integrals possess singularities of the type $1/\epsilon^L$. Except for the lowest expansion coefficient in the four-point function, and the lowest two coefficients in the two-point function, all higher expansion coefficients are convergent for $\epsilon \rightarrow 0$.

4-4 Renormalization

A. PERTURBATIVE RENORMALIZATION

(1) UV divergences:

For ϕ^4 -theories, the two-point vertex function up to order two-loop is

$$\bar{\Gamma}^{(2)}(p) = \left(\text{---} \right)^{-1} - \frac{(-\lambda)}{2} \text{---} \text{---} \text{---} - \frac{(-\lambda)^2}{4} \text{---} \text{---} \text{---} - \frac{(-\lambda)^2}{6} \text{---} \text{---} \text{---}$$

$$\bar{\Gamma}^{(2)}(p) = p^2 + m^2 - \Sigma(p)$$

$$= p^2 + m^2 + \frac{\lambda}{2} E_1(m) - \frac{\lambda^2}{4} E_1(m) E_2(m) - \frac{\lambda^2}{6} E_3(p, m) + O(\lambda^3)$$

where $E_1(m)$, $E_2(m)$, and $E_3(m)$ will diverge in ultraviolet. We can absorb these divergences into few newly defined parameters. For example, according to the regularization, the divergences of $E_1(m)$ and $E_2(m)$ are independent of p^2 . The self-energy $\Sigma(0)$ represents the dynamical contribution to the mass.

(2) Mass and coupling constant renormalization in the one-loop approximation:

At the one-loop level, we define a **renormalized mass** m_1 ,

$$m_1^2 = m^2 + \frac{\lambda}{2} E_1(m)$$

Inverting this equation gives $m(m_1)$:

$$m^2 = m_1^2 - \frac{\lambda}{2} E_1(m)$$

Since

$$\begin{aligned}
E_1(m) &= \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \\
&= \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m_1^2 - \frac{\lambda}{2} E_1(m)} \\
&= \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m_1^2} \left(1 + \frac{\lambda}{2} \frac{1}{q^2 + m_1^2} E_1(m) + O(\lambda^2) \right) \\
&= E_1(m_1) + \frac{\lambda}{2} E_2(m_1) E_1(m) + O(\lambda^2) \\
&= E_1(m_1) + \frac{\lambda}{2} E_2(m_1) E_1(m_1) + O(\lambda^2)
\end{aligned}$$

the two-loop term is created as m is replaced by m_1 in $E_1(m)$. Thus, we need to introduce a new coupling constant to cancel the additional two-loop terms.

Consider a four-point vertex function:

$$\bar{\Gamma}^{(4)}(p) = \lambda - \frac{\lambda^2}{2} (I_1(p, m) + 2 \text{ permutations})$$

where the notation "2 permutations" indicates that the function $I_1(p, m)$ contributes also with the permuted arguments $p = p_1 + p_2$, $p = p_1 + p_3$ and $p = p_1 + p_4$.

Since

$$\begin{aligned}
I_1(p, m) &= \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \frac{1}{(p - q)^2 + m^2} \\
&= \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m_1^2} \frac{1}{(p - q)^2 + m_1^2} \left(1 + \frac{\lambda}{2} \frac{1}{q^2 + m_1^2} E_1(m) \right)^2 \\
&= I_1(p, m_1) + \lambda I_2(p, m_1) E_1(m) + O(\lambda)
\end{aligned}$$

the divergence of $I_1(p, m_1)$ is independent of p , i.e., the only divergence occurs in $I_1(0, m_1)$. Thus, we define a **renormalized coupling constant** λ_1 with all external momenta set equal to zero,

$$\lambda_1 = \lambda - \frac{3\lambda^2}{2} I_1(0, m_1)$$

Inverting this equation gives $\lambda(\lambda_1)$:

$$\begin{aligned}
\lambda &= \lambda_1 + \frac{3\lambda^2}{2} I_1(0, m_1) \\
&= \lambda_1 + \frac{3}{2} \left(\lambda_1 + \frac{3\lambda^2}{2} I_1(0, m_1) \right)^2 I_1(0, m_1) \\
&= \lambda_1 + \frac{3\lambda_1^2}{2} I_1(0, m_1) + O(\lambda_1^3)
\end{aligned}$$

Now, $\frac{\lambda}{2} E_1(m)$ becomes,

$$\begin{aligned}
\frac{\lambda}{2} E_1(m) &= \frac{\lambda}{2} E_1(m_1) + \frac{\lambda^2}{4} E_2(m_1) E_1(m_1) + O(\lambda^2) \\
&= \frac{1}{2} \left(\underbrace{\lambda_1}_{\checkmark} + \underbrace{\frac{3\lambda_1^2}{2} I_1(0, m_1)}_{\checkmark} + O(\lambda_1^3) \right) E_1(m_1) \\
&\quad + \frac{1}{4} \left(\underbrace{\lambda_1}_{\checkmark} + \frac{3\lambda_1^2}{2} I_1(0, m_1) + O(\lambda_1^3) \right)^2 E_2(m_1) E_1(m_1) + O(\lambda_1^2) \\
&= \frac{\lambda_1}{2} E_1(m_1) + \frac{3\lambda_1^2}{4} I_1(0, m_1) E_1(m_1) + \frac{\lambda_1^2}{4} E_2(m_1) E_1(m_1) + O(\lambda_1^3) \\
&= \frac{\lambda_1}{2} E_1(m_1) + \frac{3\lambda_1^2}{4} E_2(m_1) E_1(m_1) + \frac{\lambda_1^2}{4} E_2(m_1) E_1(m_1) + O(\lambda_1^3) \\
&= \frac{\lambda_1}{2} E_1(m_1) + \lambda_1 E_2(m_1) E_1(m_1) + O(\lambda_1^3)
\end{aligned}$$

Thus, the bare mass can be expressed as

$$m^2 = m_1^2 - \frac{\lambda}{2} E_1(m) = m_1^2 - \frac{\lambda_1}{2} E_1(m_1) - \lambda_1 E_2(m_1) E_1(m_1) - O(\lambda_1^3)$$

The two-point vertex function becomes

$$\begin{aligned}
\bar{\Gamma}^{(2)}(p) &= p^2 + m^2 + \frac{\lambda}{2} E_1(m) \\
&= p^2 + \left[m_1^2 - \frac{\lambda_1}{2} E_1(m_1) - \lambda_1 E_2(m_1) E_1(m_1) - O(\lambda_1^3) \right] \\
&\quad + \left[\frac{\lambda_1}{2} E_1(m_1) + \lambda_1 E_2(m_1) E_1(m_1) + O(\lambda_1^3) \right] \\
&= p^2 + m_1^2 + O(\lambda_1^3)
\end{aligned}$$

The renormalized two-point vertex function is

$$\bar{\Gamma}_R^{(2)}(p) = p^2 + m_1^2$$

We obtain a renormalization condition: $\bar{\Gamma}_R^{(2)}(0) = m_1^2$.

Expressing the renormalized four-point function in terms of λ_1 , we

obtain

$$\begin{aligned}
\bar{\Gamma}_R^{(4)}(p) &= \lambda - \frac{\lambda^2}{2} (I_1(p, m) + 2 \text{ permutations}) \\
&= \left(\lambda_1 + \frac{3\lambda_1^2}{2} I_1(0, m_1) + O(\lambda_1^3) \right) \\
&\quad - \frac{1}{2} \left(\lambda_1 + \frac{3\lambda_1^2}{2} I_1(0, m_1) + O(\lambda_1^3) \right)^2 (I_1(p, m_1) + 2 \text{ perms}) \\
&= \lambda_1 + \frac{3\lambda_1^2}{2} I_1(0, m_1) - \frac{\lambda_1^2}{2} (I_1(p, m_1) + 2 \text{ permutations}) \\
&= \lambda_1 - \frac{\lambda_1^2}{2} (I_1(p, m_1) + 2 \text{ permutations} - 3I_1(0, m_1))
\end{aligned}$$

The subtracted expression is calculated for $\epsilon \rightarrow 0$,

$$\begin{aligned}
\lambda_1^2 I_1(p, m_1) &= \frac{\mu^\epsilon g_1^2}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(1) + \ln \frac{4\pi\mu^2}{m_1^2} - L(p, m_1) \right] \\
\lambda_1^2 I_1(0, m_1) &= \frac{\mu^\epsilon g_1^2}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(1) + \ln \frac{4\pi\mu^2}{m_1^2} \right]
\end{aligned}$$

where

$$L(p, m) = -2 + \sqrt{1 + \frac{4m^2}{p^2}} \ln \frac{\sqrt{1 + 4m^2/p^2} + 1}{\sqrt{1 + 4m^2/p^2} - 1}, \quad p > 0$$

We obtain

$$\begin{aligned}
&\frac{\lambda_1^2}{2} (I_1(p, m_1) + 2 \text{ permutations} - I_1(0, m_1)) \\
&= -\frac{\mu^\epsilon g_1^2}{2(4\pi)^2} (L(s, m_1) + L(t, m_1) + L(u, m_1)) \\
&= \text{finite}
\end{aligned}$$

where $s = p_1 + p_2$, $t = p_1 + p_3$, and $u = p_1 + p_4$.

The renormalized four-point vertex function is

$$\bar{\Gamma}_R^{(4)}(p) = \lambda_1 - \frac{\lambda_1^2}{2} (I_1(p, m_1) + 2 \text{ permutations} - 3I_1(0, m_1))$$

We obtain a renormalization condition: $\bar{\Gamma}_R^{(4)}(0) = \lambda_1$

(3) Field renormalization in the two-loop approximation:

The renormalization condition: $\bar{\Gamma}_R^{(2)}(0) = m_1^2$ leads to

$$m_1^2 = m^2 + \frac{\lambda}{2} E_1(m) - \frac{\lambda^2}{4} E_1(m) E_2(m) - \frac{\lambda^2}{6} E_3(0, m)$$

Inverting this equation gives $m(m_1)$:

$$\begin{aligned}
m^2 &= m_1^2 - \frac{\lambda}{2} E_1(m) + \frac{\lambda^2}{4} E_1(m) E_2(m) + \frac{\lambda^2}{6} E_3(0, m) \\
&= m_1^2 - \left[\frac{\lambda_1}{2} E_1(m_1) + \lambda_1^2 E_2(m_1) E_1(m_1) + O(\lambda_1^3) \right] \\
&\quad + \frac{\lambda_1^2}{4} E_1(m_1) E_2(m_1) + \frac{\lambda_1^2}{6} E_3(0, m_1) + O(\lambda_1^3) \\
&= m_1^2 - \frac{\lambda_1}{2} E_1(m_1) - \frac{3\lambda_1^2}{4} E_1(m_1) E_2(m_1) + \frac{\lambda_1^2}{6} E_3(0, m_1) + O(\lambda_1^3)
\end{aligned}$$

$\bar{\Gamma}^{(2)}(p)$ can now be written as:

$$\begin{aligned}
\bar{\Gamma}^{(2)}(p) &= p^2 + m_1^2 - \underbrace{\frac{\lambda_1}{2} E_1(m_1) - \frac{3\lambda_1^2}{4} E_1(m_1) E_2(m_1) + \frac{\lambda_1^2}{6} E_3(0, m_1)}_m \\
&\quad + \underbrace{\frac{\lambda_1}{2} E_1(m_1) + \lambda_1^2 E_2(m_1) E_1(m_1)}_{\text{one-loop}} \\
&\quad - \underbrace{\frac{\lambda_1^2}{4} E_1(m_1) E_2(m_1)}_{\text{two-loop}} - \underbrace{\frac{\lambda_1^2}{6} E_3(p, m_1)}_{\text{two-loop}} + O(\lambda_1^3) \\
&= p^2 + m_1^2 - \frac{\lambda_1^2}{6} \left(E_3(p, m_1) - E_3(0, m_1) \right)
\end{aligned}$$

The subtracted expression is calculated for $\epsilon \rightarrow 0$,

$$\begin{aligned}
\lambda_1^2 E_3(p, m_1) &= -\frac{m_1^2 g_1^2}{(4\pi)^4} \left[\frac{6}{\epsilon^2} + \frac{18}{2\epsilon} + \frac{6\psi(1)}{\epsilon} + \frac{6}{\epsilon} \ln \frac{4\pi\mu^2}{m_1^2} + \frac{p^2}{2m_1^2 \epsilon} \right] \\
\lambda_1^2 E_3(0, m_1) &= -\frac{m_1^2 g_1^2}{(4\pi)^4} \left[\frac{6}{\epsilon^2} + \frac{18}{2\epsilon} + \frac{6\psi(1)}{\epsilon} + \frac{6}{\epsilon} \ln \frac{4\pi\mu^2}{m_1^2} \right] \\
\lambda_1^2 \left(E_3(p, m_1) - E_3(0, m_1) \right) &= -\frac{m_1^2 g_1^2}{(4\pi)^4} \frac{p^2}{2m_1^2 \epsilon} = -\frac{g_1^2}{2(4\pi)^4} \frac{p^2}{\epsilon}
\end{aligned}$$

Thus we can expand $E_3(p, m_1)$ into the sum as,

$$E_3(p, m_1) = E_3(0, m_1) + p^2 \left. \frac{\partial}{\partial p^2} E_3(p, m_1) \right|_{p=0}$$

The subtracted expression in $\bar{\Gamma}^{(2)}(p)$ is

$$\begin{aligned}
& \frac{\lambda_1^2}{6} (E_3(p, m_1) - E_3(0, m_1)) \\
&= \frac{\lambda_1^2}{6} \left(E_3(0, m_1) + p^2 \frac{\partial}{\partial p^2} E_3(p, m_1) \Big|_{p=0} - E_3(0, m_1) \right) \\
&= \frac{\lambda_1^2}{6} p^2 \frac{\partial}{\partial p^2} E_3(p, m_1) \Big|_{p=0}
\end{aligned}$$

The two-point vertex function is

$$\begin{aligned}
\bar{\Gamma}^{(2)}(p) &= p^2 + m_1^2 - \frac{\lambda_1^2}{6} p^2 \frac{\partial}{\partial p^2} E_3(p, m_1) \Big|_{p=0} \\
&= \left(1 - \frac{\lambda_1^2}{6} \frac{\partial}{\partial p^2} E_3(p, m_1) \Big|_{p=0} \right) p^2 + m_1^2
\end{aligned}$$

Let

$$\frac{1}{Z} = 1 - \frac{\lambda_1^2}{6} \frac{\partial}{\partial p^2} E_3(p, m_1) \Big|_{p=0}$$

We obtain

$$\bar{\Gamma}^{(2)}(p) = \frac{1}{Z} (p^2 + Z m_1^2) = \frac{1}{Z} \bar{\Gamma}_R^{(2)}(p)$$

where $\bar{\Gamma}_R^{(2)}(p)$ is the renormalized two-point vertex function.

We obtain a renormalization condition: $\frac{\partial}{\partial p^2} \bar{\Gamma}_R^{(2)}(0) = 1$

- (4) At the one-loop and two-loop levels, all divergences have been removed by a redefinition of the mass, coupling constant and field associated with the renormalization conditions:

$$\begin{aligned}
\bar{\Gamma}_R^{(2)}(0) &= m_R^2 \\
\frac{\partial}{\partial p^2} \bar{\Gamma}_R^{(2)}(0) &= 1 \\
\bar{\Gamma}_R^{(4)}(0) &= \lambda_R
\end{aligned}$$

- (5) Power counting:

We can define the superficial degree of divergence of the diagram as,

$$\delta = D + n \left(1 - \frac{D}{2} \right) + p(D - 4)$$

where

D is the dimension

n is the number of external lines

p is the number of vertices

In four dimension, $D = 4$, this simplifies to

$$\delta = 4 + n \left(1 - \frac{4}{2} \right) + p(4 - 4) = 4 - n$$

For $\delta \geq 0$, the diagram is said to be divergent and Since only diagrams with $\delta \geq 0$ need renormalization, the only diagrams needing it in ϕ^4 -theory are those with $n \leq 4$. It is implying that only two- and four-point irreducible diagrams are divergent and left to renormalize.

B. RENORMALIZATION CONSTANTS

- (1) One important feature to note about these integrals is that when we differentiate them with respect to external momenta, the integral will become more convergent.

For example, if we differentiate $E_1(p, m)$ with respect to p^2 , one finds

$$\frac{\partial}{\partial p^2} E_1(p, m) = \frac{\partial}{\partial p^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(p - q)^2 + m^2}$$

Let $x = p^2, p = \sqrt{x}$

$$\begin{aligned} \frac{\partial}{\partial p^2} E_1(p, m) &= \frac{\partial}{\partial x} \int \frac{d^D q}{(2\pi)^D} \frac{1}{x - 2\sqrt{x}q + q^2 + m^2} \\ &= - \int \frac{d^D q}{(2\pi)^D} \frac{1 - \frac{q}{\sqrt{x}}}{(x - 2\sqrt{x}q + q^2 + m^2)^2} \\ &= - \int \frac{d^D q}{(2\pi)^D} \frac{\sqrt{x} - q}{\sqrt{x}(x - 2\sqrt{x}q + q^2 + m^2)^2} \\ &= - \int \frac{d^D q}{(2\pi)^D} \frac{p - q}{p \left((p - q)^2 + m^2 \right)^2} \end{aligned}$$

For $D = 4$, the integral is logarithmically divergent.

For example, if we differentiate $I_1(p, m)$ with respect to p^2 , one finds

$$\begin{aligned}
\frac{\partial}{\partial p^2} I_1(p, m) &= \frac{\partial}{\partial p^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \frac{1}{(p - q)^2 + m^2} \\
&= \frac{\partial}{\partial x} \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \frac{1}{x - 2\sqrt{x}q + q^2 + m^2} \\
&= - \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \frac{1 - \frac{q}{\sqrt{x}}}{(x - 2\sqrt{x}q + q^2 + m^2)^2} \\
&= - \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \frac{\sqrt{x} - q}{\sqrt{x}(x - 2\sqrt{x}q + q^2 + m^2)^2} \\
&= - \int \frac{d^D q}{(2\pi)^D} \frac{1}{q^2 + m^2} \frac{p - q}{p \left((p - q)^2 + m^2 \right)^2}
\end{aligned}$$

For $D = 4$, the integral is finite. This means that the divergences will reside only in the first few terms in a Taylor expansion in the external momenta of the Feynman diagram.

- (2) We expanded the self-energy in external momentum p will have 2 divergent terms,

$$\Sigma(p) = \Sigma(0) + p^2 \frac{\partial}{\partial p^2} \Sigma(p) \Big|_{p=0} + \Sigma_c(p) = \Sigma(0) + p^2 \Sigma'(0) + \Sigma_c(p)$$

where $\Sigma(0)$ is quadratically divergent and $\frac{\partial}{\partial p^2} \Sigma(p) = \Sigma'(p)$ is logarithmically divergent.

The two-point vertex function is

$$\begin{aligned}
\bar{\Gamma}^{(2)}(p) &= p^2 + m^2 - \Sigma(0) - p^2 \Sigma'(0) - \Sigma_c(p) \\
&= p^2 (1 - \Sigma'(0)) + (m^2 - \Sigma(0)) - \Sigma_c(p) \\
&= p^2 (1 - \Sigma'(0)) + m^2 \left(1 - \frac{\Sigma(0)}{m^2} \right) - \Sigma_c(p) \\
&= (1 - \Sigma'(0)) \left(p^2 + \frac{m^2 (1 - \Sigma(0)/m^2)}{1 - \Sigma'(0)} \right) - \frac{(1 - \Sigma'(0)) \Sigma_c(p)}{1 - \Sigma'(0)}
\end{aligned}$$

Defining the renormalized mass and self-energy,

$$\begin{aligned}
m_R^2 &= \frac{(1 - \Sigma(0)/m^2)}{1 - \Sigma'(0)} m^2 \\
\bar{\Sigma}(p) &= \frac{\Sigma_c(p)}{1 - \Sigma'(0)}
\end{aligned}$$

Then,

$$\bar{\Gamma}^{(2)}(p) = (1 - \Sigma'(0)) (p^2 + m_R^2 - \bar{\Sigma}(p)) = (1 - \Sigma'(0)) \bar{\Gamma}_R^{(2)}(p)$$

where

$$\bar{\Gamma}_R^{(2)}(p) = p^2 + m_R^2 - \bar{\Sigma}(p)$$

which satisfies the renormalization conditions:

$$\bar{\Gamma}_R^{(2)}(0) = m_R^2$$

$$\frac{\partial}{\partial p^2} \bar{\Gamma}_R^{(2)}(0) = 1$$

$$\bar{\Gamma}_R^{(4)}(0) = \lambda_R$$

- (3) The two-point vertex functions $\bar{\Gamma}^{(2)}(p)$ can be made finite via a multiplicative renormalization employing three renormalization constants, Z , Z_m , and Z_λ ,

$$Z = \frac{1}{1 - \Sigma'(0)}$$

$$\phi = \frac{1}{\sqrt{1 - \Sigma'(0)}} \bar{\phi} = \sqrt{Z} \bar{\phi}$$

$$Z_m = \frac{1}{1 - \Sigma(0)/m^2}$$

so that the two-point vertex function is

$$\bar{\Gamma}^{(2)}(p) = \frac{1}{Z} \left(p^2 + \frac{Z}{Z_m} m^2 - Z \Sigma_c(p) \right)$$

Let

$$m^2 = \frac{Z_m}{Z} m_R^2$$

$$\lambda = \frac{Z_\lambda}{Z^2} \lambda_R$$

Thus, we have

$$\bar{\Gamma}^{(2)}(p) = \frac{1}{Z} (p^2 + m_R^2 - \bar{\Sigma}(p)) = \frac{1}{Z} \bar{\Gamma}_R^{(2)}(p)$$

C. MINIMAL SUBTRACTION

- (1) For ϕ^4 -theories, given the Lagrangian,

$$L = \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 + \frac{\lambda}{4!} \phi^4$$

We define a renormalized field as

$$\phi = \sqrt{Z}\bar{\phi}$$

The Lagrangian can be rewritten as,

$$L = \frac{1}{2}Z(\nabla\bar{\phi})^2 + \frac{1}{2}Zm^2\bar{\phi}^2 + \frac{Z^2\lambda}{4!}\bar{\phi}^4$$

Defining

$$L_R = \frac{1}{2}(\nabla\bar{\phi})^2 + \frac{1}{2}m_R^2\bar{\phi}^2 + \frac{\lambda_R}{4!}\bar{\phi}^4$$

where m_R and λ_R are physical mass and coupling.

The renormalized Lagrangian arises by splitting up the Lagrangian in its renormalized part and the rest, the counter-terms which have absorbed the infinite

$$L = L_R + \underbrace{\frac{1}{2}(Z-1)(\nabla\bar{\phi})^2 + \frac{1}{2}(Zm^2 - m_R^2)\bar{\phi}^2 + \frac{1}{4!}(Z^2\lambda - \lambda_R)\bar{\phi}^4}_{\text{counter-terms}}$$

From the renormalization conditions,

$$m^2 = \frac{Z_m}{Z}m_R^2$$

$$\lambda = \frac{Z_\lambda}{Z^2}\lambda_R$$

we obtain

$$L_R = L - \underbrace{\frac{1}{2}(Z-1)(\nabla\bar{\phi})^2 - \frac{1}{2}\left(Z\frac{Z_m}{Z}m_R^2 - m_R^2\right)\bar{\phi}^2 - \frac{1}{4!}\left(Z^2\frac{Z_\lambda}{Z^2}\lambda_R - \lambda_R\right)\bar{\phi}^4}_{\text{counter-terms}}$$

$$= L - \frac{1}{2}(Z-1)(\nabla\bar{\phi})^2 - \frac{1}{2}(Z_m-1)m_R^2\bar{\phi}^2 - \frac{1}{4!}(Z_\lambda-1)\lambda_R\bar{\phi}^4$$

We can then define the renormalization constants,

$$Z-1 = c_z$$

$$Z_m-1 = c_m$$

$$Z_\lambda-1 = c_\lambda$$

and obtain the renormalized Lagrangian as,

$$L_R$$

$$= \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4 - \underbrace{\frac{1}{2}c_z(\nabla\bar{\phi})^2 - \frac{1}{2}c_m m_R^2\bar{\phi}^2 - \frac{c_\lambda\lambda_R}{4!}\bar{\phi}^4}_{\text{interaction}}$$

The counterterms c_z , c_m , and c_λ produce additional vertices in the diagrammatic expansion. In momentum space, these terms have the form

$$\begin{aligned}
-\bigcirc &= -c_z \\
-\times &= -c_m m_R^2 \\
\bullet &= -c_\lambda \lambda_R
\end{aligned}$$

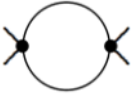
OS:

In 1929, Heisenberg and Pauli wrote down a general formulation for QFT and noted the problem of infinities in using perturbation theory. After 1947 the problem was considered solved, if there are a finite number of counter-terms needed to render perturbation theory useful, QFTs are called renormalizable.

- (2) Define a projection operator $\hat{\mathcal{K}}$ which picks up pole terms only, for example,



$$-\lambda E_1(m) = \frac{m^2 g}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(2) + \ln \frac{4\pi\mu^2}{m^2} \right]$$



$$\lambda^2 I_1(p, m) = \frac{\mu^\epsilon g^2}{(4\pi)^2} \left[\frac{2}{\epsilon} + \psi(1) + \int_0^1 dx \ln \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right) \right]$$

Apply $\hat{\mathcal{K}}$ to a diagram and we obtain

$$\hat{\mathcal{K}}(-\lambda E_1(m)) = \frac{m^2 g}{(4\pi)^2} \frac{2}{\epsilon} = m^2 \frac{g}{(4\pi)^2} \frac{2}{\epsilon}$$

$$\hat{\mathcal{K}}(\lambda^2 I_1(p, m)) = \frac{\mu^\epsilon g^2}{(4\pi)^2} \frac{2}{\epsilon} = \mu^\epsilon g \frac{g}{(4\pi)^2} \frac{2}{\epsilon}$$

They are proportional to m^2 for the quadratically divergent diagram, and to $\mu^\epsilon g$ for the logarithmically divergent diagram.

Note that the absence of the external momenta in the pole term on the right-hand side implies the important relation:

$$\hat{\mathcal{K}}(-\lambda E_1(m)) = \hat{\mathcal{K}}(\lambda^2 I_1(p, m))$$

- (3) One-loop diagrams of $\bar{\Gamma}^{(2)}$:

To first order in λ , the counterterms which are necessary to make the two-point vertex function $\bar{\Gamma}^{(2)}$ finite are also of first order in λ .

$$\bar{\Gamma}^{(2)}(p) = p^2 + m^2 - \left(\frac{1}{2} \bigcirc - \text{---}\times\text{---} + \text{---}\ominus\text{---} + \mathcal{O}(g^2) \right)$$

where the cross and the small circle on a line indicate the contribution of the mass and field counter-terms c_m and c_z .

1. Counter-term c_m :



$$-c_m^1 m^2 = -\frac{1}{2} \hat{\mathcal{K}}(\lambda E_1(m)) = -m^2 \frac{g}{(4\pi)^2} \frac{1}{\epsilon}$$

2. Counter-term c_z :



Since the counterterm contains no contribution proportional to p^2 , there is no counterterm c_z^1 to first order in λ :

$$-c_z^1 p^2 = 0$$

Choosing the counterterms in this way, the $1/\epsilon$ -pole in the one-loop diagram is canceled, and the renormalized two-point vertex function,

$$\bar{\Gamma}^{(2)}(p) = p^2 + m^2 - \left[\frac{1}{2} \bigcirc - \frac{1}{2} \mathcal{K}(\bigcirc) \right] + \mathcal{O}(g^2)$$

remain finite for $\epsilon \rightarrow 0$ up to the first order in λ .

- (4) Two-loop diagrams of $\bar{\Gamma}^{(2)}$:

$$\bar{\Gamma}^{(2)} = (-)^{-1} - \left[\frac{1}{2} \bigcirc - \text{---}\times\text{---} + \text{---}\ominus\text{---} + \frac{1}{4} \bigcirc\bigcirc + \frac{1}{2} \bigcirc\text{---}\bigcirc + \frac{1}{6} \bigcirc\text{---}\bigcirc + \frac{1}{2} \bullet\bigcirc \right]$$

1. Pole terms:



$$\frac{1}{4} \hat{\mathcal{K}}(\lambda^2 E_1(m) E_2(m)) = -\frac{m^2 g^2}{(4\pi)^4} \left[\frac{1}{\epsilon^2} + \frac{\psi(1) + \psi(2)}{2\epsilon} + \frac{1}{\epsilon} \ln \frac{4\pi\mu^2}{m^2} \right]$$



$$\frac{1}{6} \hat{\mathcal{K}}(\lambda^2 E_3(p, m)) = -\frac{m^2 g^2}{(4\pi)^4} \left[\frac{1}{\epsilon^2} + \frac{3}{2\epsilon} + \frac{\psi(1)}{\epsilon} + \frac{1}{\epsilon} \ln \frac{4\pi\mu^2}{m^2} + \frac{p^2}{12m^2\epsilon} \right]$$

OS:

The divergent term proportional to p^2 will give the first contribution to the wave function counterterm c_z .

2. Counter-terms c_m :



$$\begin{aligned}
& \frac{1}{2} \widehat{\mathcal{K}}(\lambda^2 I_1(0, m)) \\
&= \frac{\mu^\epsilon g^2}{(4\pi)^2} \left[\frac{1}{\epsilon} + \frac{\psi(1)}{2} + \frac{1}{2} \int_0^1 dx \ln \left(\frac{4\pi\mu^2}{p^2 x(1-x) + m^2} \right) \right] \\
&= \frac{m^2 c_m^1 g}{(4\pi)^2} \left[\frac{2}{\epsilon} + \frac{\psi(1)}{2} + \frac{1}{2} \int_0^1 dx \ln \frac{4\pi\mu^2}{m^2} \right] \\
&= \frac{m^2 g}{(4\pi)^2} \frac{g}{(4\pi)^2} \frac{1}{\epsilon} \left[\frac{2}{\epsilon} + \frac{\psi(1)}{2} + \frac{1}{2} \ln \frac{4\pi\mu^2}{m^2} \right] \\
&= \frac{m^2 g^2}{(4\pi)^4} \left[\frac{2}{\epsilon^2} + \frac{\psi(1)}{2\epsilon} + \frac{1}{2\epsilon} \ln \frac{4\pi\mu^2}{m^2} \right]
\end{aligned}$$

Counter-term c_λ :



$$\begin{aligned}
\frac{1}{2} \widehat{\mathcal{K}}(-\lambda E_1(m)) &= \frac{m^2 g}{(4\pi)^2} \left[\frac{1}{\epsilon} + \frac{\psi(2)}{2} + \frac{1}{2} \ln \frac{4\pi\mu^2}{m^2} \right] \\
&= \frac{m^2 c_\lambda^1 g}{(4\pi)^2} \left[\frac{1}{\epsilon} + \frac{\psi(2)}{2} + \frac{1}{2} \ln \frac{4\pi\mu^2}{m^2} \right]
\end{aligned}$$

Consider



$$-c_\lambda^1 \lambda = -\frac{3}{2} \widehat{\mathcal{K}}(\lambda^2 I_1(0, m)) = -\mu^\epsilon g \frac{3g}{(4\pi)^2} \frac{1}{\epsilon}$$

Thus, we obtain

$$\begin{aligned}
\frac{1}{2} \widehat{\mathcal{K}}(-\lambda E_1(m)) &= \frac{m^2 g}{(4\pi)^2} \frac{3g}{(4\pi)^2} \frac{1}{\epsilon} \left[\frac{1}{\epsilon} + \frac{\psi(2)}{2} + \frac{1}{2} \ln \frac{4\pi\mu^2}{m^2} \right] \\
&= \frac{3m^2 g^2}{(4\pi)^4} \left[\frac{1}{\epsilon^2} + \frac{\psi(2)}{2\epsilon} + \frac{1}{2\epsilon} \ln \frac{4\pi\mu^2}{m^2} \right]
\end{aligned}$$

We find the contributions to the counter-terms c_m and c_z up to the second order in λ :

$$\begin{aligned}
-\text{cross} + \text{dot} &= -\mathcal{K} \left[\frac{1}{2} \text{tadpole} + \frac{1}{4} \text{tadpole} + \frac{1}{6} \text{tadpole} + \frac{1}{2} \text{tadpole} + \frac{1}{2} \text{tadpole} \right] \\
&- \left[\frac{g}{(4\pi)^2} \frac{m^2}{\epsilon} + \frac{g^2}{(4\pi)^4} \left(\frac{2m^2}{\epsilon^2} - \frac{m^2}{2\epsilon} - \frac{p^2}{12\epsilon} \right) \right]
\end{aligned}$$

The pole terms proportional to m^2 extend the mass counter-terms as follows:

$$(c_m^1 + c_m^2)m^2 = m^2 \left[\frac{g}{(4\pi)^2} \frac{1}{\epsilon} + \frac{g^2}{(4\pi)^4} \left(\frac{2}{\epsilon^2} - \frac{1}{2\epsilon} \right) \right]$$

The pole terms proportional to p^2 give the second-order counterterm of the field renormalization:

$$c_z p^2 = \frac{1}{6} \widehat{\mathcal{K}}(\lambda^2 E_3(p, m)) \Big|_{m=0} = -p^2 \frac{g^2}{(4\pi)^4} \frac{1}{12\epsilon}$$

- (5) Up to the order λ^2 , we thus obtain finite correlation functions by starting out from the initial Lagrangian L , with the renormalization constants,

$$\begin{aligned} Z = 1 + c_z &= 1 + \frac{1}{p^2} \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) \Big|_{m=0} = 1 - \frac{g^2}{(4\pi)^4} \frac{1}{12\epsilon} \\ Z_m = 1 + c_m &= 1 + \frac{1}{m^2} \left[\frac{1}{2} \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) + \frac{1}{4} \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) + \frac{1}{2} \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) \right. \\ &\quad \left. + \frac{1}{2} \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) + \frac{1}{6} \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) \Big|_{p^2=0} \right] \\ &= 1 + \frac{g}{(4\pi)^2} \frac{1}{\epsilon} + \frac{g^2}{(4\pi)^4} \left(\frac{2}{\epsilon^2} - \frac{1}{2\epsilon} \right) \\ Z_g = 1 + c_g &= 1 + \frac{1}{\mu^\epsilon g} \left[\frac{3}{2} \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) + 3 \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) + \frac{3}{4} \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) \right. \\ &\quad \left. + \frac{3}{2} \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) + 3 \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) + 3 \widehat{\mathcal{K}}(\text{---}\bigcirc\text{---}) \right] \\ &= 1 + \frac{g}{(4\pi)^2} \frac{3}{\epsilon} + \frac{g^2}{(4\pi)^4} \left(\frac{9}{\epsilon^2} - \frac{3}{\epsilon} \right) \end{aligned}$$

4-5 Renormalization Group

A. THE WILSON MOMENTUM-SHELL RG

- (1) Lagrangian density of a ϕ^4 theory

$$L = \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4$$

- (2) The partition function is

$$Z = \int D\phi e^{-S[\phi]}$$

UV (ultra-violet) cut-off Λ

The Fourier modes ϕ_k vanish for suitable high momenta

$$\phi_k = 0 \text{ for } k > \Lambda$$

$$Z = \int [D\phi]_{\Lambda} e^{-S_{\Lambda}[\phi]} = \int \prod_{|k| \leq \Lambda} d\phi(k) e^{-S_{\Lambda}[\phi]}$$

Separate $\phi(k)$ into high energy part $\tilde{\varphi}(k)$ and low energy part $\varphi(k)$

$$\phi(k) = \varphi(k) + \tilde{\varphi}(k)$$

with

$$\phi(k) = \begin{cases} \varphi(k), & \text{for } 0 < |k| < b\Lambda, \\ \tilde{\varphi}(k), & \text{for } b\Lambda < |k| < \Lambda \end{cases} \quad b < 1$$

Integrate out $\tilde{\varphi}$

$$Z = \int [D\varphi][D\tilde{\varphi}] e^{-S_{\Lambda}[\varphi + \tilde{\varphi}]} = \int [D\varphi] S^{-S_{b\Lambda}[\varphi]}$$

$S_{b\Lambda}[\varphi]$ is called the Wilsonian effective action where only modes of φ with $|k| < b\Lambda$ in the action.

- (3) Including J :

$$J(k) = \begin{cases} j(k), & \text{for } 0 < |k| < b\Lambda \\ \tilde{j}(k), & \text{for } b\Lambda < |k| < \Lambda \end{cases}$$

$$\begin{aligned} Z[J] &= \frac{1}{N} \int [D\phi]_{\Lambda} e^{-S_{\Lambda}[\phi] + \int d^D k J(k) \phi(k)} \\ &= \frac{1}{N} \int [D\varphi] D[\tilde{\varphi}] e^{-S_{\Lambda}[\varphi + \tilde{\varphi}] + \int_0^{b\Lambda} d^D k j(k) \varphi(-k) + \int_{b\Lambda}^{\Lambda} d^D k \tilde{j}(k) \tilde{\varphi}(-k)} \end{aligned}$$

Let

$$e^{-W[\tilde{j}, \varphi]} = \int [D\tilde{\varphi}] e^{-S_{\Lambda}[\varphi + \tilde{\varphi}] + S_{\Lambda}[\varphi] + \int_{b\Lambda}^{\Lambda} d^D k \tilde{j}(k) \tilde{\varphi}(-k)}$$

$W[j, \varphi]$ generates connected Green's functions w.r.t. $\tilde{\varphi}$.

$$Z[J] = \frac{1}{N} \int [D\varphi] e^{-S_\Lambda[\varphi] - W[j, \varphi] + \int_0^{b\Lambda} d^D k j(k) \varphi(-k)}$$

(4) The Wilsonian effective action

$$S_{b\Lambda}[\varphi] = S_\Lambda[\varphi] + W[j, \varphi]_{j=0}$$

$S_{b\Lambda}[\varphi]$ contains original $S_\Lambda[\varphi]$ with coupling constant corrections from $W[j, \varphi]_{j=0}$ and also new interaction terms, higher point interaction and higher derivative interaction.

B. ONE-LOOP RENORMALIZATION

(1) Lagrangian of a ϕ^4 theory

$$L = \frac{1}{2} \nabla \phi \cdot \nabla \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4$$

$$\begin{aligned} S_\Lambda[\phi] &= S_\Lambda[\varphi + \tilde{\varphi}] \\ &= \int d^D x \left\{ \frac{1}{2} \nabla(\varphi + \tilde{\varphi}) \cdot \nabla(\varphi + \tilde{\varphi}) + \frac{1}{2} m^2 (\varphi + \tilde{\varphi})^2 + \frac{1}{4!} \lambda (\varphi + \tilde{\varphi})^4 \right\} \\ &= S_\Lambda[\varphi] + \tilde{S}[\varphi, \tilde{\varphi}] \end{aligned}$$

where

$$\begin{aligned} \tilde{S}[\varphi, \tilde{\varphi}] &= \int d^D x \left\{ \underbrace{\frac{1}{2} \nabla \tilde{\varphi} \cdot \nabla \tilde{\varphi}}_{\text{free part}} \right. \\ &\quad \left. + \underbrace{\frac{1}{2} m^2 \tilde{\varphi}^2 + \frac{1}{4!} \lambda \tilde{\varphi}^4 + \frac{1}{3!} \lambda (\tilde{\varphi}^3 \varphi + \tilde{\varphi} \varphi^3) + \frac{1}{4} \lambda \varphi^2 \tilde{\varphi}^2}_{\text{interaction part}} \right\} \end{aligned}$$

Integrate out $\tilde{\varphi}$

$$e^{-W[j, \varphi]} = \int [D\tilde{\varphi}]_{b\Lambda < |k| < \Lambda} e^{-\tilde{S}[\varphi, \tilde{\varphi}] - \int_{b\Lambda}^\Lambda d^D x j \tilde{\varphi}}$$

Fourier transformation in momentum space

$$\begin{aligned} \tilde{\varphi}(x) &= \int_{b\Lambda}^\Lambda \frac{d^D k}{(2\pi)^D} \tilde{\varphi}(k) e^{-ik \cdot x} \\ \int d^3 x \nabla \tilde{\varphi} \cdot \nabla \tilde{\varphi} &= \int_{b\Lambda}^\Lambda \frac{d^D k}{(2\pi)^D} \tilde{\varphi}(-k) k^2 \tilde{\varphi}(k) \\ \int d^3 x j \tilde{\varphi} &= \int_{b\Lambda}^\Lambda \frac{d^D k}{(2\pi)^D} j(k) \tilde{\varphi}(-k) \end{aligned}$$

$$\begin{aligned}
W[\tilde{j}, \varphi]_{\tilde{j}=0} &= -\ln \int [D\tilde{\varphi}]_{b\Lambda < |k| < \Lambda} e^{-\tilde{S}[\varphi, \tilde{\varphi}] + \int_{b\Lambda}^{\Lambda} \frac{d^D k}{(2\pi)^D} \tilde{j}(k) \tilde{\varphi}(-k)} \Bigg|_{\tilde{j}=0} \\
&= -\ln \left[N_0 e^{-\tilde{S}_{\text{int}} \left[\varphi, (2\pi)^D \frac{\delta}{\delta \tilde{j}(-k)} \right]_{Z_0}[\tilde{j}]} \right]_{\tilde{j}=0}
\end{aligned}$$

where

$$\begin{aligned}
\tilde{S}_{\text{int}}[\varphi, \tilde{\varphi}] &= \frac{1}{2} m^2 \tilde{\varphi}^2 + \frac{1}{4!} \lambda \tilde{\varphi}^4 + \frac{1}{3!} \lambda (\tilde{\varphi}^3 \varphi + \tilde{\varphi} \varphi) + \frac{1}{4} \lambda \varphi^2 \tilde{\varphi}^2 \\
Z_0[\tilde{j}] &= \frac{\int [D\tilde{\varphi}] e^{-S_0[\tilde{\varphi}] + \int_{b\Lambda}^{\Lambda} \frac{d^D k}{(2\pi)^D} \tilde{j}(k) \tilde{\varphi}(-k)}}{\int [D\tilde{\varphi}] e^{-S_0[\tilde{\varphi}]}} = e^{\frac{1}{2} \int_{b\Lambda}^{\Lambda} \frac{d^D k}{(2\pi)^D} \tilde{j}(-k) \frac{1}{k^2} \tilde{j}(k)}
\end{aligned}$$

(2) One-loop contribution in $W[\tilde{j}, \varphi]_{\tilde{j}=0}$

$$\begin{aligned}
\overline{\tilde{\varphi}(k) \tilde{\varphi}(p)} &= \left((2\pi)^D \frac{\delta}{\delta \tilde{j}(-k)} \right) \left((2\pi)^D \frac{\delta}{\delta \tilde{j}(-p)} \right) e^{\frac{1}{2} \int_{b\Lambda}^{\Lambda} \frac{d^D k}{(2\pi)^D} \tilde{j}(-k) \frac{1}{k^2} \tilde{j}(k)} \\
&= \frac{1}{k^2} (2\pi)^D \delta^{(D)}(k+p) \Theta(k)
\end{aligned}$$

where

$$\Theta(k) = \begin{cases} 1, & b\Lambda \leq |k| < \Lambda \\ 0, & \text{otherwise} \end{cases}$$

1. $O(g)$ terms, i.e., φ^2 term:



$$\begin{aligned}
&\frac{\lambda}{4} \int d^D x \varphi^2(x) \overline{\tilde{\varphi}(x) \tilde{\varphi}(x)} \\
&= \frac{\lambda}{4} \int d^D x \varphi^2(x) \int_{b\Lambda}^{\Lambda} \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} e^{-i(k_1+k_2) \cdot x} \overline{\tilde{\varphi}(k_1) \tilde{\varphi}(k_2)} \\
&= \frac{\mu}{2} \int d^D x \varphi^2(x)
\end{aligned}$$

where

$$\mu = \frac{\lambda}{2} \int_{b\Lambda < |k| < \Lambda} \frac{d^D k}{(2\pi)^D} \frac{1}{k^2} = \frac{g}{(4\pi)^{D/2} \Gamma(D/2)} \frac{(1 - b^{D-2})}{D-2} \Lambda^{D-2}$$

Wilsonian effective action:

$$S_{b\Lambda}[\varphi] = S_{\Lambda}[\varphi] + W[\tilde{j}, \varphi]_{\tilde{j}=0}$$

$$L_{b\Lambda} \xrightarrow{\varphi^2 \text{ term}} \frac{1}{2} \tilde{m}^2 \phi^2 = \frac{1}{2} (m^2 + \mu) \phi^2$$

There is no contribution to the kinetic term

2. $O(g^2)$ terms, i.e., φ^4 term:



$$\begin{aligned} I &= \int d^D x d^D y \frac{\lambda}{4} \phi^2(x) \tilde{\phi}(x) \tilde{\phi}(x) \frac{\lambda}{4} \phi^2(y) \tilde{\phi}(y) \tilde{\phi}(y) \\ &= \frac{\lambda^2}{16} \int d^D x d^D y \phi^2(x) \phi^2(y) \int \frac{d^D k_1}{(2\pi)^D} \frac{d^D k_2}{(2\pi)^D} e^{i(k_1+k_2) \cdot (x-y)} \frac{1}{k_1^2 k_2^2} \end{aligned}$$

Let $k = k_1, p = k + k_2$. We obtain

$$I = \frac{\lambda^2}{16} \int \frac{d^D p}{(2\pi)^D} \int d^D x d^D y e^{ip \cdot (x-y)} \phi^2(x) \phi^2(y) \int_{b\Lambda}^{\Lambda} \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 (p-k)^2}$$

Since

$$\frac{1}{k^2 (p-k)^2} = \frac{1}{k^4} \left[1 + O\left(\frac{p^2}{k^2}\right) \right] \approx \frac{1}{k^4}$$

we obtain

$$\begin{aligned} I &= \frac{\lambda^2}{16} \int d^D x d^D y \phi^2(x) \phi^2(y) \delta^{(D)}(x-y) \int_{b\Lambda}^{\Lambda} \frac{d^D k}{(2\pi)^D} \frac{1}{k^4} \\ &= -\frac{\zeta}{4!} \int d^D x \phi^4(x) \end{aligned}$$

where

$$\zeta = -\frac{4! \lambda^2}{16} \int_{b\Lambda}^{\Lambda} \frac{d^D k}{(2\pi)^D} \frac{1}{k^4} = -\frac{3\lambda^2}{(4\pi)^{D/2} \Gamma(D/2)} \frac{(1 - b^{D-4})}{D-4} \Lambda^{D-4}$$

Wilsonian effective action:

$$\begin{aligned} S_{b\Lambda}[\varphi] &= S_{\Lambda}[\varphi] + W[\tilde{j}, \varphi]_{\tilde{j}=0} \\ L_{b\Lambda} &\rightarrow \frac{1}{4!} \tilde{\lambda} \phi^4 = \frac{1}{4!} (\lambda + \zeta) \phi^4 \end{aligned}$$

C. RENORMALIZATION GROUP

(1) We start off from a renormalizable theory,

$$L = \frac{1}{2} \nabla \phi \nabla \phi + \frac{1}{2} m^2 \phi^2 + \frac{1}{4!} \lambda \phi^4 + C (\nabla \phi)^4 + D \phi^6 + \dots$$

After integrate out $b\Lambda < |k| < \Lambda$ modes

$$L = \frac{1}{2}Z\nabla\varphi\nabla\varphi + \frac{1}{2}\tilde{m}^2\varphi^2 + \frac{1}{4!}\tilde{\lambda}\varphi^4 + \tilde{C}(\nabla\varphi)^4 + \tilde{D}\varphi^6 + \dots$$

where

$$Z = 1 + \Delta Z$$

$$\tilde{m}^2 = m^2 + \Delta m^2$$

$$\tilde{\lambda} = \lambda + \Delta\lambda$$

$$\tilde{C} = C + \Delta C$$

$$\tilde{D} = D + \Delta D$$

- (2) To compare the new functional integral and the old one. We can rescale the cut-off $b\Lambda$ to Λ .

$$k' = \frac{k}{b}, \quad x' = bx, \quad |k'| < \Lambda$$

$$\begin{aligned} & \int d^d x L_{b\Lambda} \\ &= \int d^d x' b^{-3} \left[\frac{1}{2} Z b^2 \nabla' \varphi \nabla' \varphi + \frac{1}{2} \tilde{m}^2 \varphi^2 + \frac{1}{4!} \tilde{\lambda} \varphi^4 + \tilde{C} b^4 (\nabla' \varphi)^4 + \tilde{D} \varphi^6 \right. \\ & \left. + \dots \right] \end{aligned}$$

rescale the φ to give the same propagator as the original theory:

$$\begin{aligned} & \varphi'(x) = [b^{-1}Z]^{1/2} \varphi \\ & \int d^d x L_{b\Lambda} \\ &= \int d^d x' \left[\frac{1}{2} Z b^2 \nabla' \varphi' \nabla' \varphi' + \frac{1}{2} m'^2 \varphi'^2 + \frac{1}{4!} \lambda' \varphi'^4 + C' (\nabla' \varphi')^4 + D' \varphi'^6 \right. \\ & \left. + \dots \right] \end{aligned}$$

We have defined the new parameters:

$$Z = 1 + \Delta Z$$

$$m'^2 = (m^2 + \Delta m^2) Z^{-1} b^{-2}$$

$$\lambda' = (\lambda + \Delta\lambda) Z^{-2} b^{d-4}$$

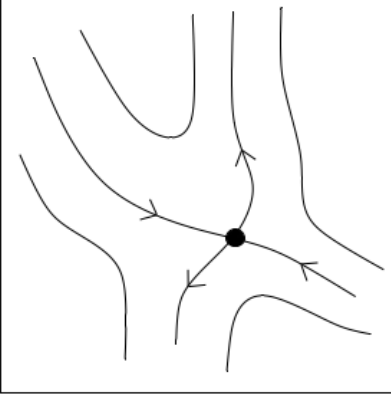
$$C' = (C + \Delta C) Z^{-2} b^d$$

$$D' = (D + \Delta D) Z^{-3} b^{2d-6}$$

The form of the Lagrangian is the same as the original one, with only parameters changed - a transformation of the Lagrangian.

- (3) Repeat this procedure step by step, we integrate out higher momentum shell and take $b \rightarrow 1$, an infinitesimal thin shell. The transformations of

parameters become continuous - a trajectory or a flow in the space of all possible Lagrangian - Renormalization group.



By continuous RG evolution from Λ , iterating the integrate-out procedure, all the high energy effects are absorbed slowly and systematically into the coupling constants - as long as the constants remain small.

D. MARGINAL, RELEVANT, IRRELEVANT

- (1) A little away from zero-coupling to one-loop.

Let $\Delta Z = 0$

$$m'^2 = (m^2 + \Delta m^2)Z^{-1}b^{-2}$$

$$\Delta m^2 = \frac{\lambda}{(4\pi)^{d/2}\Gamma(d/2)} \frac{(1 - b^{d-2})}{d-2} \Lambda^{d-2}$$

$$\lambda' = (\lambda + \Delta\lambda)Z^{-2}b^{d-4}$$

$$\Delta\lambda = -\frac{3\lambda^2}{(4\pi)^{d/2}\Gamma(d/2)} \frac{(1 - b^{d-4})}{d-4} \Lambda^{d-4}$$

Consider an infinitesimal change

$$b = 1 + \frac{\Delta\Lambda}{\Lambda}, \quad \Delta\Lambda < 0$$

Define

$$\begin{aligned} \tilde{m}^2 &= \frac{m^2}{\Lambda^2}, & \tilde{m}'^2 &= \frac{m'^2}{\Lambda'^2} = \frac{(m^2 + \Delta m^2)Z^{-1}}{\Lambda'^2} \\ \tilde{\lambda} &= \frac{\lambda}{\Lambda^{4-d}}, & \tilde{\lambda}' &= \frac{\lambda'}{\Lambda'^{4-d}} = \frac{(\lambda + \Delta\lambda)Z^{-2}}{\Lambda'^{4-d}} \end{aligned}$$

Dimensionless parameters:

$$\Lambda \frac{d\tilde{m}^2}{d\Lambda} = -\frac{\tilde{\lambda}}{(4\pi)^{d/2}\Gamma(d/2)} - 2\tilde{m}^2$$

$$\Lambda \frac{d\tilde{\lambda}}{d\Lambda} = \frac{3\tilde{\lambda}^2}{(4\pi)^{d/2}\Gamma(d/2)} + \tilde{\lambda}(d-4)$$

(2) Trivial solutions

$$\Delta Z \approx \Delta m^2 \approx \Delta \lambda \approx \Delta C \approx \Delta D \approx \dots \approx 0$$

$$\phi^2, \quad m'^2 = m^2 b^{-2}$$

$$\phi^4, \quad \lambda' = \lambda b^{-1}$$

$$(\nabla\phi)^4, \quad C' = C b^3$$

$$\phi^6, \quad D' = D$$

Behavior of the running of the constants around

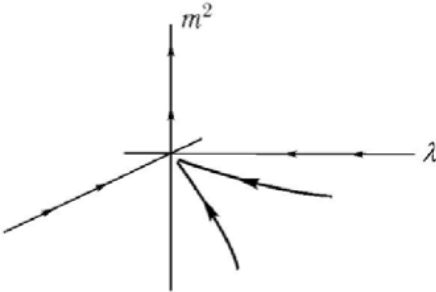
$$m^2 = \lambda = C = D = \dots = 0$$

the free fixed point.

(3) $d > 4$

$$\Lambda \frac{d\tilde{m}^2}{d\Lambda} < 0 \Rightarrow \Lambda \rightarrow 0, \tilde{m}^2 \uparrow \Rightarrow m^2 \text{ relevant}$$

$$\Lambda \frac{d\tilde{\lambda}}{d\Lambda} > 0 \Rightarrow \Lambda \rightarrow 0, \tilde{\lambda} \downarrow, \Rightarrow \lambda \text{ irrelevant}$$



The initial conditions for the RG flow are adjusted so that the trajectory passes very close to the free fixed point. The low energy theory will be a free theory.

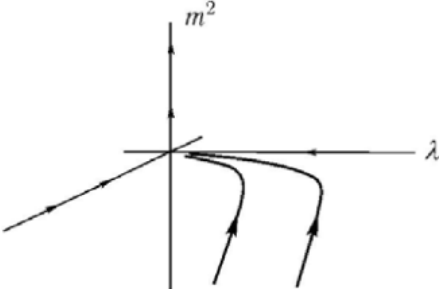
(4) $d = 4$

$$\Lambda \frac{d\tilde{\lambda}}{d\Lambda} = \frac{3\tilde{\lambda}^2}{16\pi^2} > 0 \Rightarrow \Lambda \downarrow, \tilde{\lambda} \downarrow \Rightarrow \lambda \text{ marginal}$$

Alternative,

$$\tilde{\lambda} = \frac{\tilde{\lambda}_0}{1 - \frac{3\tilde{\lambda}_0}{16\pi^2} \ln\left(\frac{\Lambda}{\Lambda_0}\right)} \Rightarrow \Lambda \rightarrow 0, \tilde{\lambda} \rightarrow 0 \Rightarrow \lambda \text{ marginal irrelevant}$$

$$\Lambda \frac{d\tilde{m}^2}{d\Lambda} \stackrel{\tilde{\lambda} \rightarrow 0}{\approx} -2\tilde{m}^2 < 0, \tilde{m}^2 \rightarrow \tilde{m}_0^2 \left(\frac{\Lambda_0}{\Lambda}\right)^2 \Rightarrow \Lambda \downarrow, \tilde{m}^2 \uparrow \Rightarrow m^2 \text{ relevant}$$



(5) $d < 4$

$[\lambda] = 4 - d > 0$, relevant

$$\tilde{\lambda} \rightarrow 0 \Rightarrow \Lambda \frac{d\tilde{\lambda}}{d\Lambda} \stackrel{\tilde{\lambda} \rightarrow 0}{\approx} -\tilde{\lambda}(4 - d) < 0 \Rightarrow \Lambda \downarrow, \tilde{\lambda} \uparrow \Rightarrow \lambda \text{ relevant}$$

Alternative

$$\tilde{\lambda} \approx \tilde{\lambda}_0 \left(\frac{\Lambda_0}{\Lambda}\right)^{4-d}$$

As $\tilde{\lambda}$ increases, $\Lambda \frac{d\tilde{\lambda}}{d\Lambda} \rightarrow 0$, may exhibit new fixed point (Wilson-Fisher fixed point)

$$\left\{ \begin{array}{l} \tilde{\lambda} > \tilde{\lambda}_*, \quad \Lambda \frac{d\tilde{\lambda}}{d\Lambda} > 0, \quad \Lambda \downarrow \tilde{\lambda} \downarrow \\ \tilde{\lambda} < \tilde{\lambda}_*, \quad \Lambda \frac{d\tilde{\lambda}}{d\Lambda} < 0, \quad \Lambda \downarrow \tilde{\lambda} \uparrow \end{array} \right\} \text{ approaching } \tilde{\lambda}_*$$

$$\left\{ \begin{array}{l} \tilde{m}^2 > \tilde{m}_*^2, \quad \Lambda \frac{d\tilde{m}^2}{d\Lambda} < 0, \quad \Lambda \downarrow \tilde{m}^2 \uparrow \\ \tilde{m}^2 < \tilde{m}_*^2, \quad \Lambda \frac{d\tilde{m}^2}{d\Lambda} > 0, \quad \Lambda \downarrow \tilde{m}^2 \downarrow \end{array} \right\} \text{ away from } \tilde{m}_*^2$$

